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SOLUTION OF A PROBLEM OF OPTIMAL CONTROL  
UTILIZING PONTRYAGIN'S MAXIMUM PRINCIPLE

THEODORE A. ALMSTEDT

and

DOUGLAS B. GIBSON

SOLUTION OF A PROBLEM OF  
OPTIMAL CONTROL  
UTILIZING  
PONTRYAGIN'S MAXIMUM PRINCIPLE

\* \* \* \* \*

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by

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Submitted in partial fulfillment of  
the requirements for the degree of

MASTER OF SCIENCE  
IN  
ELECTRICAL ENGINEERING

United States Naval Postgraduate School  
Monterey, California

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## ABSTRACT

The problem of obtaining a control policy which is optimum in some respect for a specified system is one of the more pressing problems of automatic control theory today. L. S. Pontryagin stated his Maximum Principle in 1956 and L. I. Rozonoer discussed and extended the work in a series of articles published in 1959.

The Maximum Principle, as it applies to the "free right end" problem for a non-linear, one-dimensional system, is utilized in conjunction with the CDC 1604 digital computer to obtain optimal control policies for several different cost functions.

The authors wish to express their appreciation for the encouragement given them by Dr. Harold A. Titus of the U. S. Naval Postgraduate School and to express their gratitude to Miss Mary E. Haynes of the Computer Facility, U. S. Naval Postgraduate School whose patient programming assistance materially aided in the successful completion of this thesis.

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## 1. Introduction.

One of the most important problems of automatic control theory is the problem of creating systems which are optimal in some prescribed sense. Since the advent of satellites and space travel, a set of problems of the type to be discussed here have taken on greater importance. Since many of the most interesting problems in this field have system equations for which the methods of the Calculus of Variations fail, new mathematical methods are required for their solution.

Suppose that the system to be investigated may be described by an equation of the form

$$\frac{dx}{dt} = G(X, U), \quad X(0) = C \quad (1-1)$$

where  $X(t)$  is the state variable and  $U(t)$  is the control variable.\* The problem then is to determine  $U(t)$  such that the system is optimized according to some specified criterion or cost function.

Two types of process are of particular importance:

- (1) A desire for  $X(t)$  to remain as close to a prescribed path as possible throughout the duration of the process and,
- (2) A desire for the final value of  $X(t)$  to be a prescribed value or in a prescribed state (Terminal Control).

---

\*Eqn. (1-1) is written in one dimension and this investigation will be confined to systems of the first order. However, all methods and techniques employed may easily be extended to n-th order systems where  $X$  and  $U$  would be replaced by the vectors  $\underline{X}$  and  $\underline{U}$ .

The type of problem as set forth in case (1) is the type investigated here and is the "Free Right End" problem as described by L. I. Rozonoer in Section I of his paper dealing with Pontryagin's Maximum Principle /1/.

A non-linear system was hypothesised and methods of obtaining a control policy which would provide an optimum system according to some specified criterion were investigated.

The system chosen for the investigation may be described by a non-linear differential equation of the first order which has the following functional form:

$$\dot{X} = f(X, U) \quad , \quad X(0) = C \quad (1-2)$$

in which one may consider  $X$  as representing the error or deviation from a prescribed path and  $U$  as representing the control effort applied. The problem was to determine the control policy  $U$ , such that some criterion function of the form

$$J = \int_0^T G(X, U) dt \quad (1-3)$$

would be minimized. Such functions as described by Eqn. (1-3) with the constraint of Eqn. (1-2), which evaluate the performance of a system are commonly referred to in the literature as "criterion" or "cost" functions. Sometimes they can be expressed simply in terms of integrals. At best, the choice of a cost function  $J$ , is a compromise between a desired criterion of goodness for the control design and one which leads to a more tractable mathematical analysis.

The specific system investigated is described by the first order, non-linear differential equation

$$\dot{X} = AX + B \sin + CU \quad (1-4)$$



with the constants having the following values:

$$A = -0.1$$

$$B = +1.0$$

$$C = +.25$$

Several different cost functions, subject to the constraint of Eqn. (1-4) were proposed for the investigation:

$$J = \int_0^T (X^2 + U^2) dt \quad (1-5)$$

$$J = \int_0^T (X^2) dt \quad (1-6)$$

$$J = \int_0^T (U^2) dt \quad (1-7)$$

The problem of obtaining a control policy to effect the minimization of these criteria (each of which leads to a different set of adjoint equations) will be treated in detail below.

Qualitatively, minimizing Eqn. (1-5) would correspond to minimizing the error and the control effort to the system. Minimizing Eqn. (1-6) would correspond to minimizing only the error, while minimizing Eqn. (1-7) would minimize only the control effort of the system with no regard for the error.

As mentioned above, this investigation deals with the "Free Right End" problem. As a result, the time interval over which the integrals, Eqs. (1-5), (1-6) and (1-7), were evaluated was fixed at 10 seconds.

By the very nature of the formulation of the problem (minimization of the integral  $J = \int_0^T G(X,U) dt$  with the constraint  $\dot{X} = f(X,U)$ , having  $X(0)$  specified, but not specifying the value of  $X$  at the final time  $T$ ), it falls within the province of the Calculus of Variations. Occasionally such classical methods can be employed to determine the optimal control policies.

The authors however, attempted to apply the work of a Russian control

theorist, L. S. Pontryagin, as it is presented in a series of articles by L. I. Rozonoer /1,2,3/. With the aid of these papers and the CDC 1604 digital computer, satisfactory control policies were obtained.

Briefly recapitulating, the purpose of this thesis is to use the digital computer, employing various programming methods, to search for, compute, and design an optimal control which will satisfy the various criteria (cost functions) as applied to Eqn. (1-4).

## 2. The uncontrolled system.

The uncontrolled system as described by the following equation

$$\dot{X} = -0.1X + \sin X \quad (2-1)$$

was investigated first. A knowledge of the behavior of the system without restraint or control applied was deemed necessary in order that investigations of the system with control could be properly interpreted and understood.

The Donner Analog computer (Model 3100) was employed in order to obtain the phase plane plot of velocity ( $\dot{X}$ ) versus trajectory ( $X$ ). Fig. (2-1) shows the computer diagram used to obtain the phase plane desired.

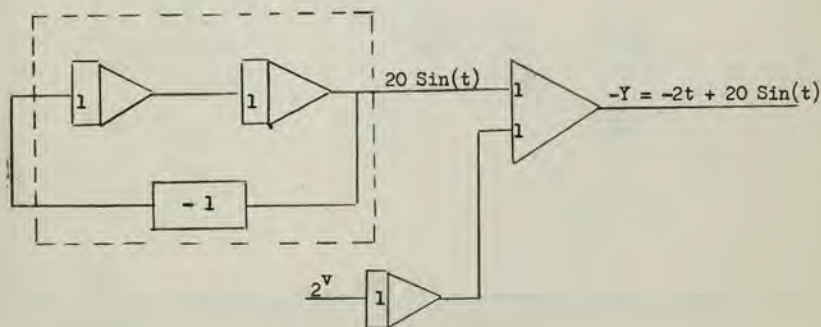


FIG. (2-1). Computer diagram utilized in order to obtain the Phase Plane ( $\dot{X}$  vs.  $X$ ) for the uncontrolled system.

As no sine function generator was available, the sine function was simulated as shown within the dotted rectangle.\*

\*The sine function was simulated by constructing the analog of the second order differential equation  $\ddot{X} + X = 0$  whose solution is known to be  $X = \sin(t)$ .

The output

$$-Y = 20 \sin t - 2t \quad (2-2)$$

was recorded on an X-Y Plotter utilizing the real time variable input to the X coordinate. Fig. (2-2) shows the results of this investigation, the phase plane plot for the uncontrolled system.

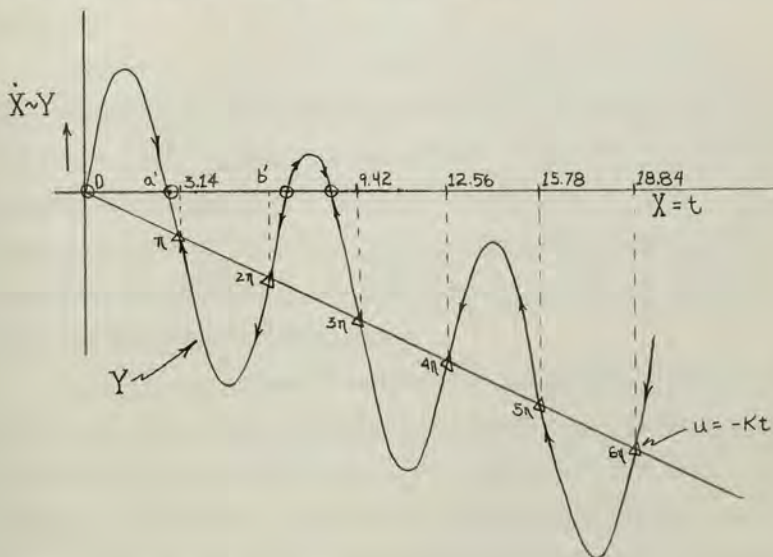


FIG. (2-2). Phase Plane Plot of the uncontrolled system showing Velocity ( $\dot{X}$ ) on the ordinate plotted against Trajectory ( $X$ ) on the abscissa.

Since the computer output was in terms of  $Y$  and Time, it was necessary to convert these coordinates to the desired coordinates  $\dot{X}$  and  $X$ . In order to scale the abscissa it was necessary to find the point on the curve corresponding to  $\pi$  radians. Since the curve as described by Eqn. (2-2) is a linear combination of a sine curve and a straight line

$$U = -Kt \quad (2-3)$$



the points where the resultant curve intersects the straight line are multiples of  $\pi$ . The points on the abscissa lying directly above these points of intersection are, therefore, also multiples of  $\pi$  radians. In this manner the X coordinate was transformed from a time scale to a distance scale. (Since only qualitative results as to velocity were desired the Y axis was not rescaled and no transformation of coordinates was made).

Investigation of Fig. (2-2) yielded the following interesting information concerning the existence of equilibrium points: There are two stable equilibrium points located at  $X = 2.84+$  and at  $X = 8.41+$  and there are two unstable equilibrium points located at  $X = 0.0$  and at  $X = 7.02+$ . Thus, the uncontrolled system may be expected to move toward one of the stable equilibrium points depending on the initial value of X as indicated by the arrows on the curve of Fig. (2-2).

In addition to the above investigation, the CDC 1604 digital computer was employed as an additional method of determining the equilibrium points. By choosing selected values of initial X (both positive and negative) and solving the differential equation (by the Runge-Kutta method), equilibrium points were obtained which agreed favorably with those obtained by the graphical analysis of the analog solution. (Fig. (I-2) and Fig. (I-3) in Appendix I are the computer graph solutions obtained from this investigation. Tabular output also was obtained but is not included here due to the volume obtained and due to the relative unimportance of this sort of detail).

In order to obtain values of the equilibrium points to an accuracy greater than that possible by either of the preceding two methods, standard mathematical methods were utilized. The derivative was set

equal to zero and the values of the equilibrium points were obtained to three decimal point accuracy with the aid of sine tables, a desk calculator and iterative procedures. The results obtained again corresponded to and verified the earlier results, yielding stable equilibrium points at  $X = 2.852+$  and at  $X = 8.416+$  with two unstable equilibrium points at  $X = 0.00$  and  $X = 7.068+$ . Only positive equilibrium points were calculated as this investigation was to be limited to positive values of  $X$ .

Interpreting the significance of the sum of these investigations, it is possible to define the regions of stability and instability for the uncontrolled system as shown below in Fig. (2-3)

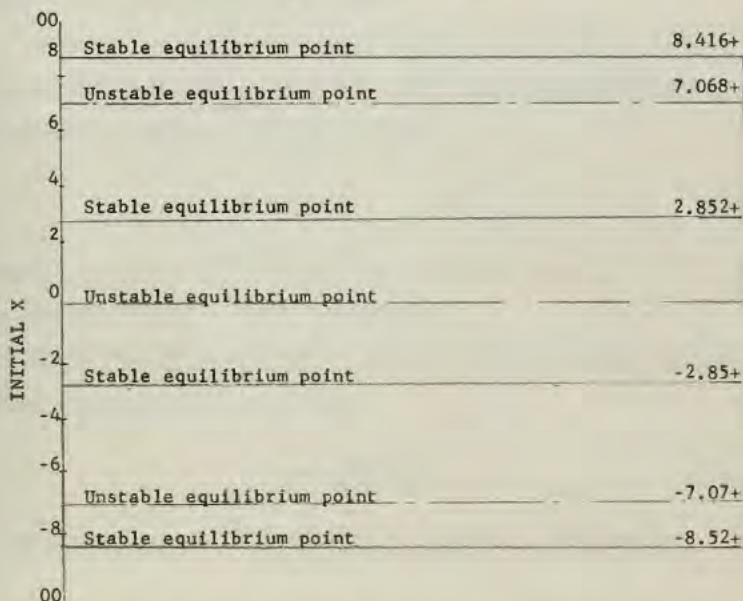


FIG. (2-3) Regions of stability and instability for the uncontrolled system.

### 3. Maximum principle in optimal system theory.

As stated above, in an effort to obtain a solution to the problem of obtaining optimal controls, the authors utilized the "Maximum Principle" as hypothesized in 1956 by L. S. Pontryagin on the basis of work performed by him, V. G. Boltyanskii and R. V. Gamkrelidze. /4/ Pontryagin's Maximum Principle is presented in some detail by L. I. Rozonoer in a series of articles appearing in "Automatika i Telemekhanika" in 1959. ("Automation and Remote Control" presents an English translation of this Russian Journal. /1.2.3/). A brief resume of the most important concepts will be made here in order that a common ground for further discussion may be established.

Rozonoer's papers deal with the problem of obtaining a control which is optimum in some sense for a system which may be described by a set of differential equations of the n-th order:\*

$$\dot{X}_i = f(X_i, U_i, t) \quad i = 1, \dots, n \quad (3-1)$$

where  $X_1, \dots, X_n$  are the parameters of the system and  $U_1, \dots, U_r$  are the positions of the controlling elements.

Rozonoer shows in his paper that the problem of optimizing the system with respect to an integral

$$J = \int G(X, U) dt \quad (3-2)$$

leads to the problem of optimizing with respect to coordinates.\*\* At this

---

\*Any n-th order differential equation may be expressed in terms of n first order differential equations involving n variables.

\*\*L. I. Rozonoer, L. S. Pontryagin's Maximum Principle in the Theory of Optimum Systems, "Automation and Remote Control", Vol 20, p1291.



point an additional variable may be introduced

$$\dot{X}_{n+1}(t) = \int_0^t F(X_1, \dots, X_n; U_1, \dots, U_r; t) dt : X_{n+1}(0) = 0 \quad (3-3)$$

which allows one more differential relation

$$\dot{X}_{n+1} = F(X_1, \dots, X_n; U_1, \dots, U_r; t) \quad (3-4)$$

to be added to the system specified in Eqn. (3-1). The problem thus becomes one of optimizing the  $n+1$ st system coordinate at the final moment of time.

Specifically, the problem of optimizing a linear function of the final values of all the coordinates of the system, that is, the quantity

$$S = \sum_{k=1}^n C_k X_k(T) \quad (3-5)$$

where  $C_k$  are certain constants, is developed in detail in Rozonoer's paper. In the discussion of this problem, the theory is first developed for the case in which the right end of the trajectory,  $X(t)$ , is not fixed. That is, there are no restrictions imposed on the final values of the coordinates.\*

A variable vector,  $P(t) = P_1(t), \dots, P_n(t)$ , which has a direction at time  $t = T$  opposite to the direction of the vector  $C = C_1, \dots, C_n$  is introduced at this point

$$P_i(T) = -C_i \quad i = 1, \dots, n \quad (3-6)$$

It is assumed that the moduli of the vectors  $\underline{P}(T)$  and  $\underline{C}$  are equal.

---

\*For the entire problem considered here, the final coordinate was left free. Only the duration of the problem was fixed at 10 seconds.

The variables  $P_i(t)$  are subject to a set of differential equations

$$\dot{P}_i(t) = - \sum_{s=1}^n P_s \frac{\partial f_s(X_1, \dots, X_n; U_1, \dots, U_r; t)}{\partial X_i} ; i = 1, \dots, n \quad (3-7)$$

(Let us note here that if any control,  $U(t)$ , is given, the vector  $\underline{P}(t)$  is uniquely determined from Eqs. (3-7), where conditions Eqs. (3-6) play the role of boundary conditions.) From Eqs. (3-5) and (3-6) boundary conditions for the final values of the adjoint variables,  $P_i$ , may be obtained.

Rozonoer continues, and points out that if the following expression is formed,

$$H(X, P, U, t) = \sum_{s=1}^n P_s \dot{X}_s \quad (3-8)$$

that Eqs. (3-2) and (3-7) may be written in the form

$$\dot{X}_i = \frac{\partial H}{\partial P_i} , \quad \dot{P}_i = - \frac{\partial H}{\partial X_i} \quad i = 1, \dots, n \quad (3-9)$$

Since the  $X_i(0)$  are specified in the problem statement and the  $P_i(T)$  may be found from Eqs. (3-6) and (3-5), the boundary conditions on Eqn. (3-9) are

$$X_i(0) = X_i^0 , \quad P_i(T) = - C_i \quad i = 1, \dots, n \quad (3-10)$$

The H-function is analogous to the Hamiltonian in analytical mechanics, and the vector  $\underline{P}(t)$  to the impulse vector. Rozonoer proves in his paper that, according to the maximum principle, the H-function must be maximum (minimum) in  $U$  for any values of  $X$  and  $P$  in order to obtain the optimum condition.

Now well known principles of Calculus are resorted to and the partial derivative of the H-function is taken with respect to  $U$  and set

equal to zero.

$$\frac{\partial H}{\partial U} = 0 \quad (3-11)$$

Eqn. (3-11) yields a relation between the control  $U$  and the adjoint variable  $P$ .

Since the cost function to be minimized has been defined as a linear combination of the final values of the coordinates of the object system, Eqn. (3-5), and this function added to the system as the  $n+1$ st coordinate, it may be seen from Eqn. (3-9) that

$$\dot{P}_{n+1} = 0 \quad (3-12)$$

in all cases. This allows immediate integration of  $\dot{P}_{n+1}$  yielding

$$P_{n+1} = \text{CONSTANT} \quad (3-13)$$

Since  $P_{n+1}$  is constant and we know from Eqn. (3-10) that  $P_{n+1}(T) = -C_{n+1}$ , we see immediately that

$$P_{n+1}(t) = -C_{n+1} \quad (3-14)$$

Also note that  $C_{n+1}$  will always be equal to one and that

$$C_i = 0 \quad i = 1, 2, \dots, n$$

Thus we see that  $P_{n+1} = -1$  which allows some simplification in Eqs. (3-9).

As a result of the above manipulations,  $2n-1$  differential equations are obtained, the solution of which yield the desired optimum control.

4. The cost function  $J = \int_0^T (X^2 + U^2) dt$ .

The first system investigated was one in which the integral

$$J = \int_0^T (X^2 + U^2) dt \quad (4-1)$$

was to be minimized. The system to be controlled was described by the differential equation

$$\dot{X} = AX + B \sin X + CU \quad (4-2)$$

for which the constants A, B, and C had the same values as specified in Section 2; namely, -0.1, 1.0, and 0.25 respectively. The time interval T, was fixed at ten seconds. (Some investigation as to the effect of extending the time interval to 20 seconds was investigated and is discussed below.)

Since the investigation of the uncontrolled system showed several points of stable and unstable equilibrium for initial values of X between zero and 10, an initial value of X equal to 20 was chosen in order to include the effects of the equilibrium points.

The problem was approached according to the principles as set forth in Rozonoer's paper, /1/, dealing with Pontryagin's Maximum Principle.

In order to apply Pontryagin's Maximum Principle, it is necessary that the system be described by an equation of the form specified by Eqn. (3-1); i.e., a first order differential equation, either vector or scalar. Since the specific system under investigation, Eqn. (4-2), is of first order, no additional manipulation was required.

Then, in accordance with Rozonoer's statement that optimizing with respect to an integral leads to the problem of optimizing with respect to



the final values of the coordinates, the cost function was added as the n+1st coordinate

$$X_2 = \int_0^T (X_1^2 + U^2) dt \quad (4-3)$$

The system equations now have the form

$$\begin{aligned} \dot{X}_1 &= AX_1 + B \sin X_1 + CU \\ \dot{X}_2 &= X_1^2 + U^2 \end{aligned} \quad (4-4)$$

Next, forming the final value functional S, Eqn. (3-5),

$$S = \sum_{i=1}^{n+1} C_i X_i(T) = C_2 X_2(T) = \int_0^T (X_1^2 + U^2) dt \quad (4-5)$$

the values of the constants  $C_i$  are obtained,

$$\begin{aligned} C_1 &= 0 \\ C_2 &= 1 \end{aligned} \quad (4-6)$$

and, since we know from Eqn. (3-6) that the adjoint variables are equal to but of opposite sense to the  $C_i$  at the final time T, we now have the boundary conditions

$$\begin{aligned} P_1(T) &= 0 \\ P_2(T) &= -1 \end{aligned} \quad (4-7)$$

Forming the H-Function, Eqn. (3-9),

$$H(P, X, U, t) = \sum_{i=1}^n P_i f_i = \sum_{i=1}^n P_i \dot{X}_i \quad (4-8)$$

and making appropriate substitutions for the  $\dot{X}_i$  the H-function becomes

$$H = P_1 (AX_1 + B \sin X_1 + CU) + P_2 (X_1^2 + U^2) . \quad (4-9)$$

From Eqs. (3-9) and (4-9) it is now possible to obtain the adjoint equations below:

$$\begin{aligned}\dot{P}_1 &= -(A + B \cos X_1)P_1 - 2X_1P_2 \\ \dot{P}_2 &= 0\end{aligned}\tag{4-10}$$

Applying the principles of Calculus to the H-function in order to obtain the minimum with respect to the control U, a relation between the adjoint variables  $P_1$  and the control variables is found.

$$\frac{\partial H}{\partial U} = CP_1 + 2UP_2 = 0 \quad U = \frac{CP_1}{2P_2}\tag{4-11}$$

Noting that the equation for  $\dot{P}_2$ , Eqn. (4-10), is readily integrable, it is seen that  $P_2(t)$  is equal to a constant. Since  $P_2(T) = -1$ , Eqn. (4-7), it follows that  $P_2$  must equal -1 at all times. Knowing the value of  $P_2(t)$ , it is possible to simplify Eqn. (4-11).

$$U = \frac{P_1}{8}\tag{4-11a}$$

Finally, making the appropriate substitutions for the constants A, B, and C and for the adjoint variable  $P_2(t)$  and replacing U by  $P_1/8$ , the following set of differential equations are obtained:

$$\dot{X}_1 = -.1X_1 + \sin X_1 + .03125 P_1\tag{4-12}$$

$$\dot{X}_2 = X_1^2 + \frac{P_1^2}{64}\tag{4-13}$$

$$\dot{P}_1 = -(.1 + \cos X_1)P_1 + 2X_1\tag{4-14}$$

The solution of the above set of equations is the problem remaining at this point.

Numerical methods and the CDC 1604 digital computer were utilized in order to obtain a solution to this problem. Specifically, the Runge-Kutta method of evaluating first order differential equations, as programmed for the CDC 1604, was employed. (See Appendix V).

In order to obtain solutions to the set of differential equations by numerical methods, initial conditions are required. Since initial conditions are known only for the X variables, ( $X_1(0)=20$ ,  $X_2(0)=0$ ), and not for the adjoint variable  $P_2$ , the immediate problem becomes one of searching the adjoint space for the initial conditions which will yield the desired solution.\*

The first method proposed in an effort to obtain  $P^*(0)$  was one in which several  $P(0)$  would be selected in an effort to bracket  $P^*(0)$ . Then, through some selective iterative scheme, the bracket size would be reduced and, eventually, the  $P^*(0)$  yielding THE OPTIMUM system could be obtained. (This method of solution is commonly called a "hill climbing" technique). There was, however, no evidence that the variation of the cost function,  $J$ , versus the initial values of  $P$  would be smooth, and the possibility of "homing in" on some local minimum rather than the true minimum was present. (The investigation of the system showed that local minima did exist in many cases.)

In view of the above, the authors decided on another approach to the problem of obtaining  $P^*(0)$ . A reasonable range of values for  $P^*(0)$  was guessed. Then, the differential equations for the system were evaluated for  $X_1(0) = 20$ ,  $X_2(0) = 0$ , and  $P_1(0)$  ranging from -250 to +50. This investigation yielded values of cost function for 300 integral values of  $P(0)$  as shown in Fig. (4-1).

A brief comment on the details of programming this problem might be

---

\*It should be pointed out here that any solution generated from Eqs. (4-12), (4-13) and (4-14) will be optimal with respect to the initial conditions chosen. However, of all these optimal solutions, there is one "best" solution. The  $P(0)$  which yields THE OPTIMUM solution will be designated  $P^*(0)$ . (The asterisk when applied to any other variable will likewise refer to THE OPTIMUM system.)



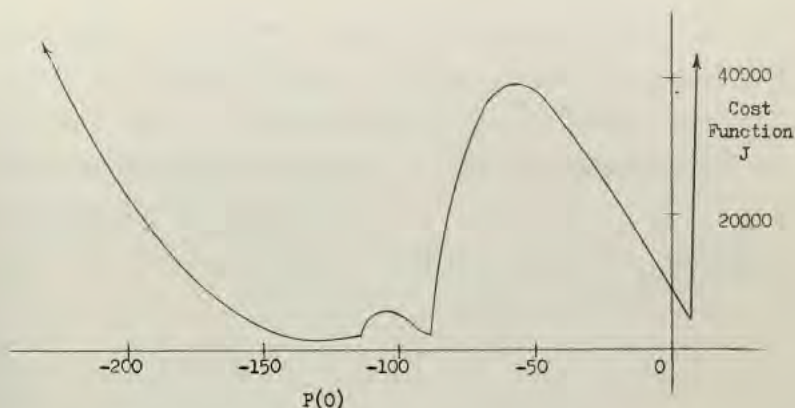


FIG. (4-1). Cost function vs. Initial value of P.  
 $X(0) = 20$   $J = \int (X^2 + U^2) dt$

appropriate here. Since it had been decided to obtain a  $P^*(0)$  by calculating the cost function for several  $P(0)$ , the next question to resolve was in regard to the range of values of  $P(0)$  to investigate. It was suggested that a first guess might be that the initial control,  $U(0)$ , should be at least as large as  $X(0)$  but of opposite sign. Utilizing Eqn. (4-11a) and recalling that  $X(0) = 20$ , a first guess puts  $P(0)$  at about -160. In an effort to allow for some uncertainty and also to obtain information as to the behavior of the cost function for  $P(0)$  not in the vicinity of  $P^*(0)$ , a range of  $P(0)$  was selected from -250 to +50 as stated above. Fig. (4-1) was the result of this initial investigation. (Not shown in Fig. (4-1), but obtained in a tabular output were the final values of the trajectory,  $X(T)$ .)

Locating the  $P^*(0)$  which yields the minimum cost function and then repeating the above investigation in the neighborhood of the first determined  $P^*(0) \pm \epsilon$  allows  $P^*(0)$  to be determined to any degree of accuracy desired.

The major disadvantage of the above method is the time required to obtain  $P^*(0)$  to a reasonable accuracy. In excess of 11 minutes was required for the above investigation. (It was not possible to obtain trajectories or control policies during this first investigation due to storage limitations of the computer.) At that, the resulting  $P^*(0)$  was accurate only to an integral value.

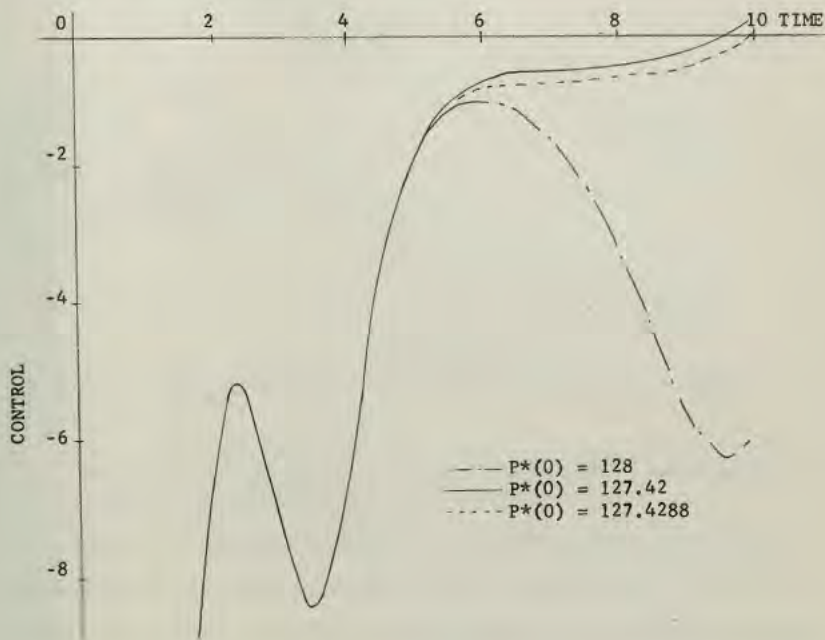


FIG. (4-2). Optimal control policies,  $U(t)$ , showing variation with the accuracy to which  $P^*(0)$  is computed for  $J = \int (X^2 + U^2) dt$ .

It should be noted that the accuracy to which  $P^*(0)$  was determined affected both the control policy and the trajectory. Fig. (4-2) shows the effect on control that the accuracy of the  $P^*(0)$  had.

The OPTIMAL Control Policy obtained from the above computations

yielded a cost function of 1099.52 versus a cost function of 1978.09 for the uncontrolled system, an improvement\* of 44.8%.

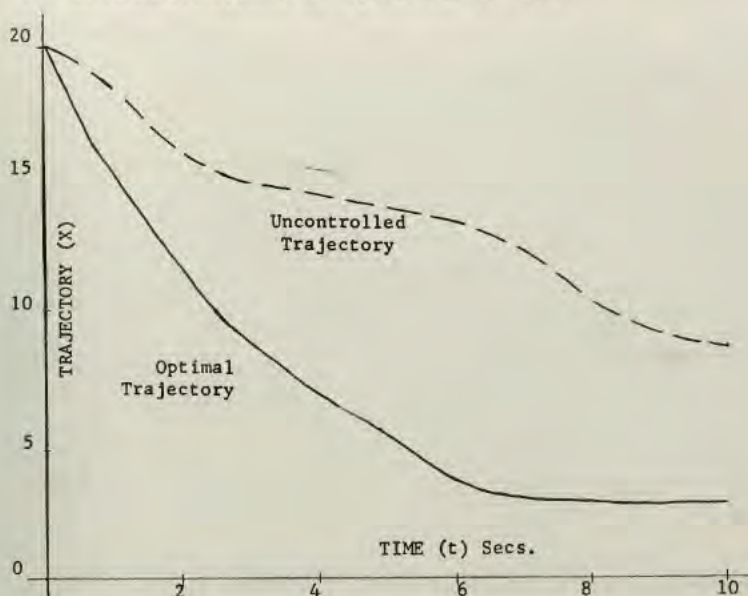


FIG. (4-3). Trajectory vs. Time for the OPTIMAL system (solid curve) and the uncontrolled system (dotted curve).

Fig. (4-3) shows the trajectories obtained for the uncontrolled system and the optimally controlled system. Note that the trajectory obtained for the optimal case as shown in Fig. (4-3) appears to approach a constant value of about 2.7 rather than a final value of zero. However, recall that the cost function to be minimized here is a combination of error,  $X$ , and control effort,  $U$ , and also that the uncontrolled system had a stable equilibrium point at  $X = 2.84$ . One might assume, therefore,

\*Percent improvement is defined as

$$\text{Percent improvement} = \frac{J(\text{uncontrolled}) - J(\text{controlled})}{J(\text{uncontrolled})}$$

that the effort necessary to reduce the error below this value would exceed any reduction in the cost function that would result from such a reduction in  $X$ .

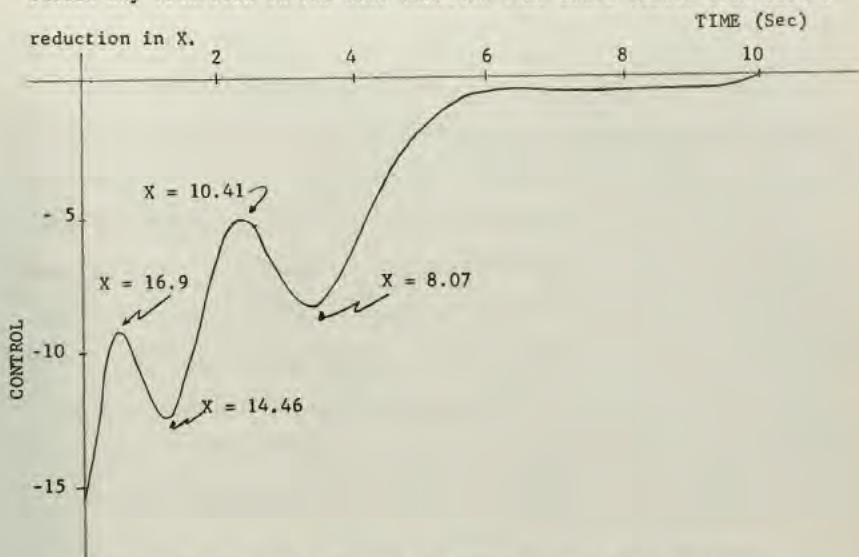


FIG. (4-4). Optimal control policy for  $X(0) = 20$  showing local maxima and minima.

Fig. (4-4) shows the optimal control policy obtained for this system with  $X(0) = 20$ . It is very interesting to note that the points of local maxima and minima occur when the trajectory, Fig. (4-3), has a value such that the velocity,  $\dot{X}$ , of the system is also very close to a maximum or minimum. (See Fig. (2-2) which is the velocity-displacement phase plane for the uncontrolled system).

Having investigated the system with the initial value of  $X$  set at 20, the next part of the investigation involved the determination of the effect of the initial value of  $X$  on the  $P^*(0)$ , the optimum control policies and the optimum trajectories. Consequently, the systems having initial values of  $X$  equal to 15, 10, and 5 were investigated. The  $P^*(0)$  in each



case was obtained in a similar manner to that described for the system in which  $X(0) = 20$ . Optimal controls and optimal trajectories were obtained in each case.

Upon investigation of the resulting optimal control policies, it was noted that they were remarkably similar in appearance to the optimal control policy for the system having an initial  $X$  equal to 20. Fig. (4-5) shows the control policies for the four systems ( $X(0) = 20, 15, 10, 5$ ). Note that the correspondence is very good for  $X(0) = 10$  and 15 until the latter part of the common time interval at which point the three policies begin to separate. Of course, this correspondence only holds if the origin of the time axis for the systems having initial  $X$  equal to 10 and 15 are moved to the right as indicated. Notice, however, that the control policy for the system having an initial  $X$  equal to 5 does not seem to correspond at all.

In spite of this latter result, the question arose as to whether the optimal control policy for the  $X(0) = 20$  system, in conjunction with its trajectory, might provide the  $P^*(0)$  for all initial values of  $X$  between 20 and the lower limit of the trajectory of about 2.7.

Consequently, systems having initial values of  $X$  of 17.5, 12.5, 7.5, 2.5, and 1.0 were investigated for  $P^*(0)$ . Having obtained the  $P^*(0)$  for these systems, the calculated  $P^*(0)$  were plotted on a curve of optimal control versus optimal trajectory for the system having initial  $X$  equal to 20, Fig. (4-6). One immediately notices the close correspondence except for those system having initial  $X$  equal to 5, 2.5, and 1.0. Since the optimal trajectory for the  $X(0) = 20$  system never got below a value of about 2.7, it is reasonable to assume that it would not be possible to

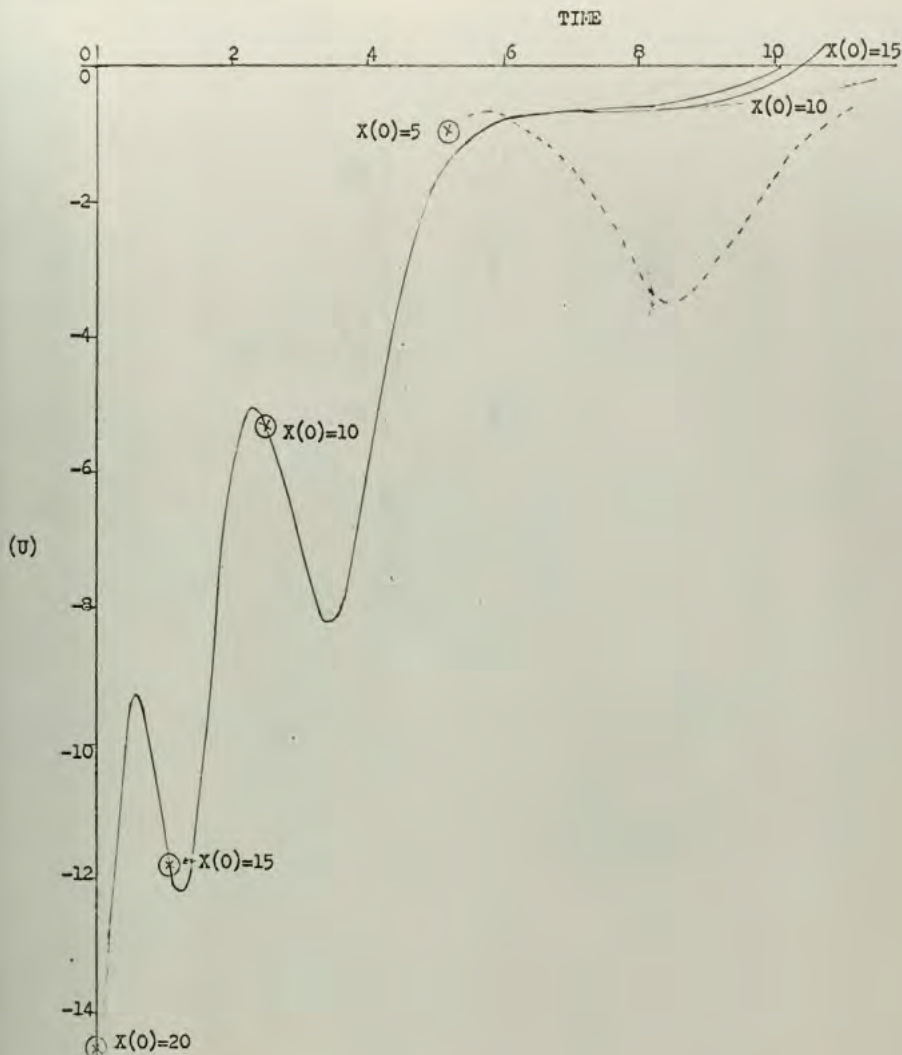


FIG. (4-5) Control vs. Time for various initial  $X$  showing correspondence except during the latter part of the common time interval and showing the non-correspondence for initial  $X$  equal to 5.0.

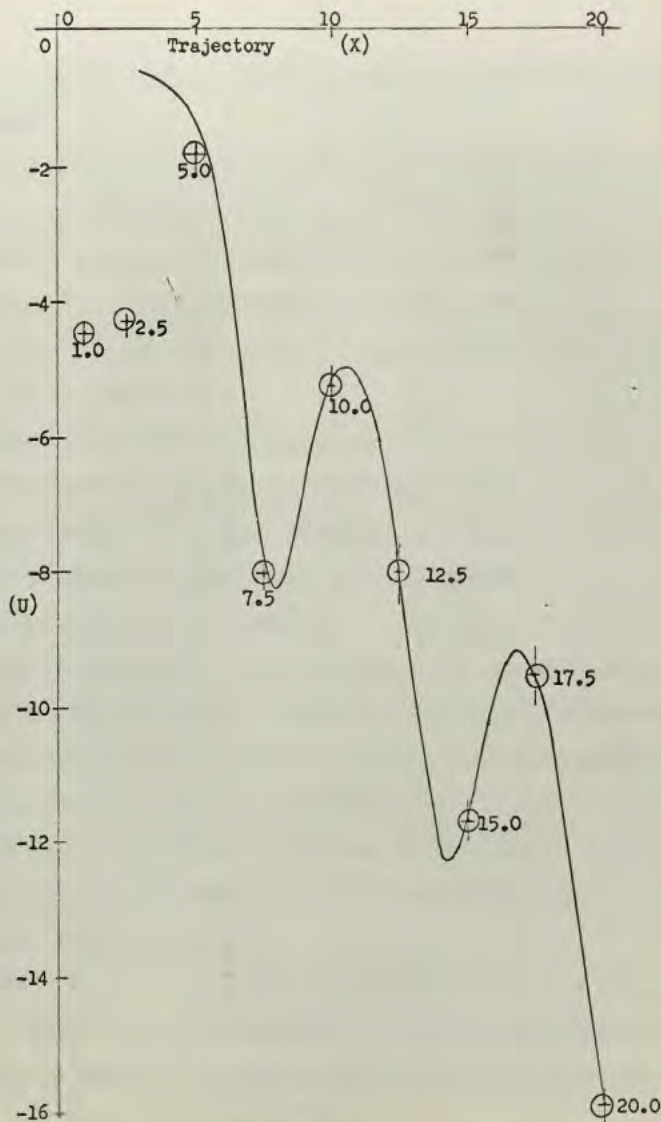


FIG.(4-6). Optimal Control plotted against Optimal Trajectory for the system having initial  $X$  equal to 20. Indicated points are the  $P^*(0)$  obtained for systems having initial  $X$  equal to the values indicated. For  $T = 10$  secs.

obtain a  $P^*(0)$  for initial  $X$  which were less than this value, i.e., not on the curve.

As a result of the above investigation, the possibility was suggested that if the system having initial  $X$  equal to 20 were allowed to run for 20 seconds and a new optimal control policy and trajectory obtained for this system, that it might be that the control policy for the  $X(0) = 5$  system would match and that this new  $U^*$  vs,  $X^*$  would provide the  $P^*(0)$  for the values of  $X$  less than 2.7.

The  $X(0) = 20$  system was allowed to run for 20 seconds and a new optimum control and trajectory were obtained as well as a plot of  $U^*$  vs.  $X^*$ . When the  $P^*(0)$  for the system investigated above were plotted on the new  $U^*$  vs  $X^*$  curve, Fig. (4-7), very close correspondence was noted in all cases including the ones for initial  $X$  of 5, 2.5 and 1.0. One must keep in mind that by extending the time interval to 20 seconds the problem became a new problem entirely. Even so, the results of this investigation show that the  $U^*$  vs.  $X^*$  curve for  $X(0) = 20$  over the extended time interval may be used to obtain an initial guess for the  $P^*(0)$  for values of  $X(0)$  between 20 and zero. Modification of the initial guess in order to obtain the true  $P^*(0)$  would be a much easier problem than the problem of obtaining  $P^*(0)$  from scratch.

The results, as stated above and as shown in the various figures, prove conclusively that it is possible to obtain numerical solutions to the problem of obtaining an optimal solution to the non-linear system by means of Pontryagin's Maximum Principle.

It is to be pointed out that a very distressing feature of this method of solution is the excessive amount of time involved in obtaining a satisfactory solution to the problem compared with the problem time



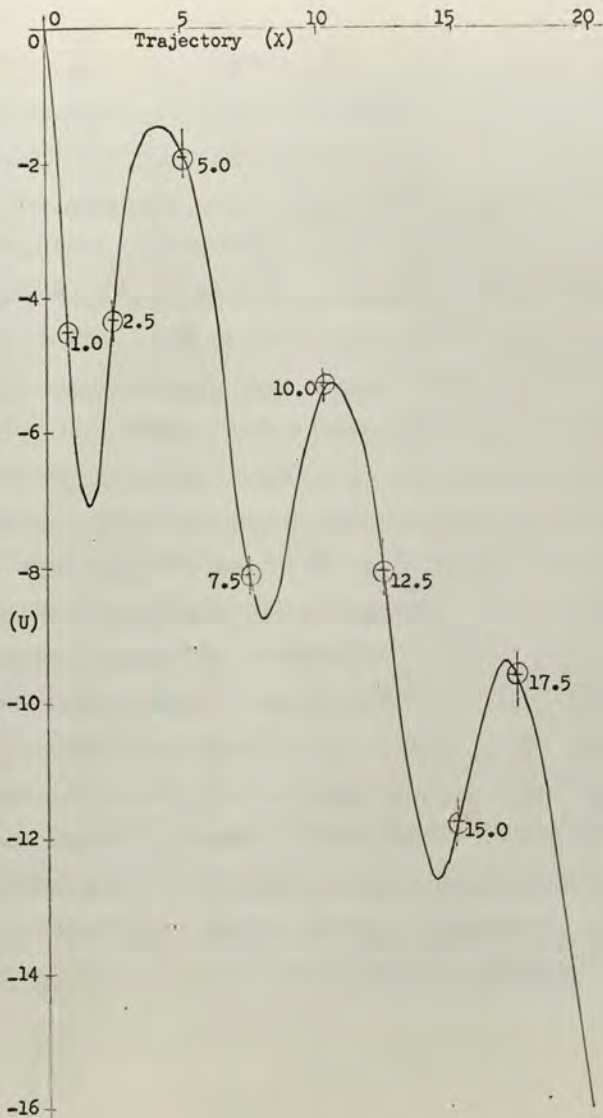


FIG.(4-7). Optimal Control plotted against Optimal Trajectory for the system having initial  $X$  equal to 20. The time interval was extended in this case to 20 seconds. Indicated points are the  $P^*(0)$  obtained for the systems having initial  $X$  equal to the values indicated.

of 10 or 20 seconds. (This would suggest some investigation into the possibility that more sophisticated programming of the CDC 1604 digital computer might result in a shorter solution time on the computer.)

Finally, one additional fact should be borne in mind. Although the digital computer was utilized successfully to obtain the optimal control function (specified at intervals of 0.1 seconds), building a physical component to duplicate this control is quite another matter. Some compromise would probably be necessary. Perhaps a square pulse of  $U$  (as in the output of a zero order hold in a sample data system) corresponding in width or modulus to the peaks shown in Fig. (4-5) would suffice for engineering purposes. Another possibility would be to feed some percentage of the value of the trajectory back to the controller (negative feedback).

In any case, each type of control would have to be evaluated and the cost function compared to the optimum one. It then becomes a question of deciding whether or not the results are sufficiently "optimum" for the engineering purpose in mind. Perhaps the only utility of an investigation such as this is in providing the "ideal" with which the engineer may judge the performance of the system he has designed.

(Additional graphs of the optimal control policies and the optimal trajectories for the above mentioned initial  $X$  supplementing and supporting the above data and conclusions may be found in Appendices I and II.)

5. The cost function  $J = \int_0^T U^2 dt$ .

The second system investigated was one in which the integral

$$J = \int_0^T U^2 dt \quad (5-1)$$

was to be minimized with the same object equation as in Section 4.

$$\dot{X} = AX + B \sin X + CU \quad (5-2)$$

Proceeding in the same manner as in Section 4, introduce the variables

$$\begin{aligned} X_1 &= X(t) \\ X_2 &= \int_0^T U^2 dt \end{aligned}$$

Substituting these variables in to Eqs. (5-1) and (5-2), and using the differential form yields the system equations

$$\dot{X}_1 = AX_1 + B \sin X_1 + CU \quad (5-3)$$

$$\dot{X}_2 = U^2 \quad (5-4)$$

By forming the final value functional

$$S = X_2(T) = \sum_{i=1}^2 C_i X_i = C_1 X_1 + C_2 X_2 \quad (5-5)$$

the C vector is obtained. In order to minimize the final state of the adjoint equations, recall that  $P_i(T)$  must equal to  $-C_i$ . Thus the following boundary conditions for the adjoint space are obtained

$$\begin{aligned} P_1(T) &= -C_1 = 0 \\ P_2(T) &= -C_2 = -1 \end{aligned} \quad (5-6)$$

Forming the H-function as before

$$H = P_1(AX_1 + B \sin X_1 + CU) + P_2 U^2 \quad (5-7)$$



and performing the partial differentiations as defined by Eqs. (3-9), the following set of simultaneous equations is obtained,

$$\begin{aligned}\dot{X}_1 &= AX_1 + B \sin X_1 + CU \\ \dot{X}_2 &= U^2 \\ \dot{P}_1 &= -(A + B \cos X_1)P_1 \\ \dot{P}_2 &= 0\end{aligned}\tag{5-8}$$

By differentiating H with respect to U and setting equal to zero, a relation between the control U, and the adjoint variables is obtained,

$$U = -\frac{CP_1}{2P_2}\tag{5-9}$$

Integrating the last of Eqs. (5-8), one is able to evaluate  $P_2(t)$  as before,

$$P_2 = \text{Constant.}$$

And since

$$P_2(T) = -1$$

then

$$P_2(t) = -1$$

Now substituting values for  $P_2$  and C into Eqn. (5-9), the relation between  $P_1$  and U becomes

$$U = \frac{P_1}{8}\tag{5-9a}$$

Now Eqs. (5-8) may be arranged in a form suitable for solution

$$\dot{X}_1 = -.1X_1 + \sin X_1 + .03125 P_1\tag{5-10}$$

$$\dot{X}_2 = P_1^2 / 64\tag{5-11}$$

$$\dot{P}_1 = (.1X_1 - \cos X_1)P_1\tag{5-12}$$

having the following boundary conditions:

$$\begin{aligned} \bar{x}_1(0) &= 20 & r_1(T) &= 0 \\ \bar{x}_2(0) &= 0 & p_2(T) &= -1 \end{aligned} \quad (5-13)$$

As in Section 4, two of the above equations, Eqs. (5-10) and (5-12), must be solved simultaneously after finding the  $P^*(0)$  which minimizes the cost function, Eqn. (5-1). But, before stating the results of this investigation, recall that the integral to be minimized is  $\int_0^T u^2 dt$ . Even without Pontryagin's Maximum Principle, the solution is immediately obvious.  $U$  must equal zero if the cost function is to be minimized, since the control  $U$ , is independent of  $X$  in the object equation, Eqn. (5-1). Thus, one might hope for a reliable check on the answers obtained by the methods utilized in this paper in the solution to this system.

Proceeding as in Section 4, values of  $P(0)$  between - 250 and + 50 were investigated and the corresponding cost functions computed. The results verified the expected minimum of zero for  $P^*(0) = 0.0$ . Fig. (5-1) shows the minimum cost function for an initial  $P$  equal to zero which resulted in a zero control for the entire time interval; i.e. the uncontrolled system. As was stated above, this is a reasonable and expected solution.

Utilizing the criteria  $J = \int_0^T u^2 dt$  provided a check on the solutions obtained in synthesizing an optimal system by the maximum principle. The results obtained from the solution of Eqs. (5-10), (5-11) and (5-12) verified the predictions made through inspection of the  $H$ -function and its derivative, Eqs. (5-7) and (5-9).

It appears from the above discussion, that this is not a very sensible way to use  $\int u^2 dt$  as a cost function. A more meaningful use would be to

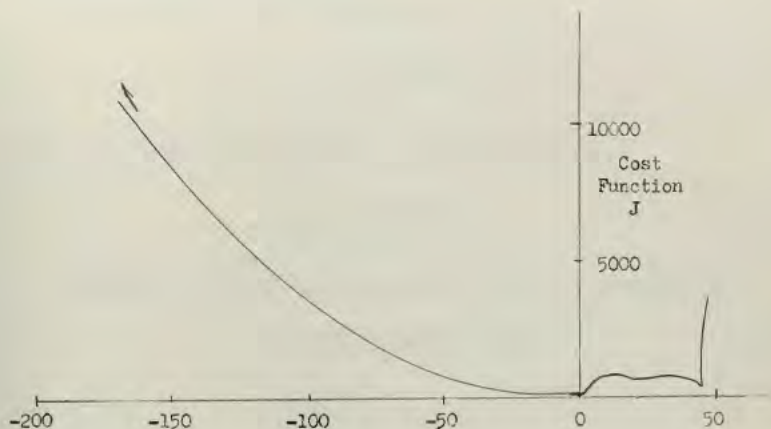


FIG. (5-1). Cost Function vs. Initial Value of  $P$ .  $X(0) = 20$ .  
 $J = \int (U^2) dt$ , showing a minimum cost function  $J = 0$ , at  $P(0) = 0$ .

apply the above cost function in a problem where the state of the trajectory at time  $(T) = 10$  secs. is fixed, say  $X(T) = 0.0$ .

That is, to even have a reasonable problem we must have a fixed right end trajectory and also fix time,  $(T)$ .

6. The cost function  $J = \int_0^T \dot{X}^2 dt$ .

The third system to be investigated was one in which the integral

$$J = \int_0^T \dot{X}^2 dt \quad (6-1)$$

was to be minimized, again with the object equation

$$\dot{X} = AX + B \sin t + CU \quad (6-2)$$

Proceeding as before, it is seen that the system equations are

$$\dot{X}_1 = AX_1 + B \sin X_1 + CU \quad (6-3)$$

$$\dot{X}_2 = X_1^2$$

Forming the H-function

$$H(P, X, U, t) = P_1(AX_1 + B \sin X_1 + CU) + P_2 X_1^2 \quad (6-4)$$

and taking suitable differentials, Eqs. (3-9), it is possible to obtain the adjoint equations. Four simultaneous equations are thus obtained which, when solved, provide the solution to the problem of obtaining the optimal system.

$$\begin{aligned} \dot{X}_1 &= AX_1 + B \sin X_1 + CU \\ \dot{X}_2 &= X_1^2 \\ \dot{P}_1 &= -P_1(A + B \cos X_1) + 2P_2 X_1 \\ \dot{P}_2 &= 0 \end{aligned} \quad (6-5)$$

From the final value functional, Eqn. (3-5),

$$S = \sum C_i X_i = C_1 X_1 + C_2 X_2 = \int_0^T \dot{X}_1^2 dt \quad (6-6)$$

the C vector is obtained. As before, by setting  $P_i(T)$  equal to  $-C_i$ , the terminal boundary conditions for the  $P_i(t)$  are obtained.



$$P_1(T) = 0 \quad (6-7)$$

$$P_2(T) = -1$$

Since the last of Eqs. (6-5) may be integrated, it is seen that

$$P_2(t) = \text{CONSTANT} \quad (6-8)$$

and, since

$$P_2(T) = -1$$

it is seen that

$$P_2(t) = -1 \quad (6-9)$$

Substituting this value for  $P_2$  into the H-function, and then differentiating with respect to U and setting the resultant relation equal to zero in order to find the minimum, results in

$$P_1 C = 0 \quad (6-10)$$

Since C is a constant equal to .25, the obvious conclusion must be that

$$P_1 = 0 \quad (6-11)$$

At first the last results seem quite disheartening as no relation between U and the adjoint space exist. However, this might have been anticipated since the H-function is linear in U and, as such, could have a minimum only at one of the boundaries.

Reconsidering the Principle of the Maximum in view of the above results, one realizes that in order to minimize the H function, Eqn. (6-4), for any X and P, U must be as large as possible and of opposite sign to X (since the constant C is positive and it is desired to make that term negative). But, since most practical systems have an upper

maximum on the amount of control available,  $U$  would then have to assume the largest value possible under such a constraint.\* In mathematical notation, the above result would be expressed as

$$U = - \operatorname{Sgn} X \mid U \mid_{\max} \quad (6-12)$$

As a result of the above developments, the only system equation needed to obtain the optimal control for this system is

$$\dot{X}_1 = AX_1 + B \sin X_1 - \operatorname{Sgn} X_1 \mid U \mid_{\max} \quad (6-13)$$

Eqn. (6-13) was solved numerically on the digital computer by means of the Runge-Kutta method for evaluation of Differential Equations. A computer solution was desired in order to verify the conclusions made on the basis of the above results and knowledge of the uncontrolled system, namely; (1) The cost function should decrease with increasing values of Control  $U$ , and (2), Once the system approached and was driven through zero, a chatter mode should be obtained for the remainder of the interval.

The initial computer solutions immediately verified that the Cost Function did decrease as the absolute magnitude of the Control was increased. However, the chatter mode did not appear in the form expected. Fig. (6-1) shows that for a Control having a magnitude of 25, the Trajectory approaches zero in about 2.8 seconds and then begins to increase for a short while and then is driven back towards the zero point. This variation of the trajectory with time was not the chatter mode expected.

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\*This result is similar to the well known Bang-Bang principle. Using Pontryagin's theory, this would be equivalent to minimizing  $\int dt$  used as the final value functional.

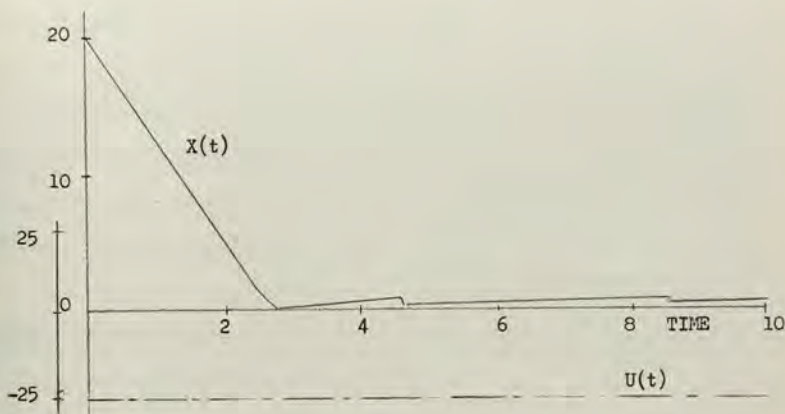


FIG. (6-1). Trajectory,  $X(t)$ , and Control,  $U(t)$  plotted as a function of time, showing the chatter mode initially obtained from the solution of the  $\int \ddot{x}^2 dt$  system.

Note also that the control remained constant at -25 for the entire interval. The authors had expected that the negative control would drive the trajectory past the zero point, change sign as  $X(t)$  became negative according to Eqn. (6-12), and then drive  $X(t)$  positive. The cycle should repeat itself until the end of the interval. These results were not obtained as may be seen in Fig. (6-1).

It was suggested that perhaps the sampling rate (0.1 sec) might be too coarse to show the chatter and that "noise" was interfering with the computations. Consequently, a sampling rate of .025 seconds was tried, but with no success. Results similar to those shown in Fig. (6-1) were obtained.

At this point, the program being utilized in the solution of this system came under close scrutiny. It was discovered that the method

used to evaluate the differential equation (Runge-Kutta) utilized the last  $X(t)$  computed in the interval  $dt$  to set the sign of  $U(t)$  for the next evaluation of  $X(t)$  in the same interval. The program should have been using the sign of the  $X(t)$  computed during the  $n$ -1st time interval to control the sign of  $U(t)$  during the  $n$ th time interval. Because of the nature of the Runge-Kutta routine, four values of  $X(t)$  were being averaged in each time interval, some being positive values and some possibly negative, resulting in the erroneous results.

The program was revised to correct the above deficiency and the results obtained showed the chatter mode and the alternating nature of the control as expected. See Fig. (6-2).

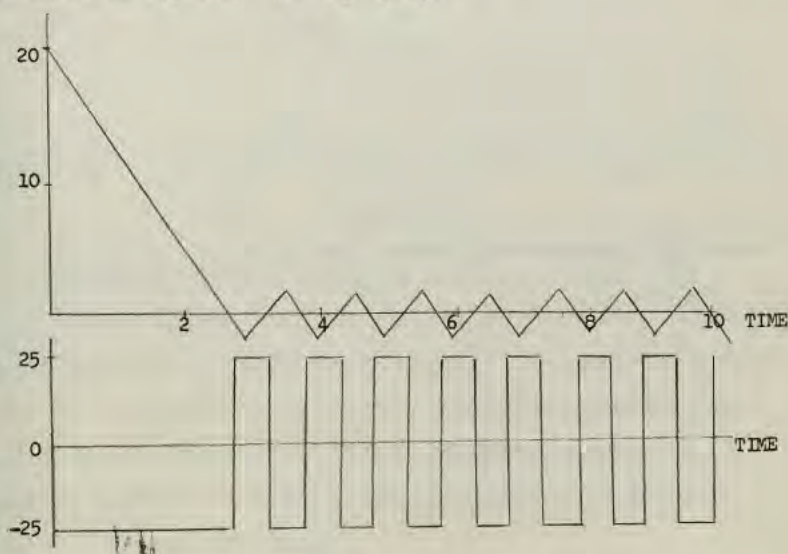


FIG. (6-2). Sketch showing the chatter mode for  $X(t)$  and the alternating characteristics of  $U(t)$  for the system having  $\int X^2 dt$  as a cost function.

The results obtained using the modified program confirmed the earlier



results that the cost function decreased with increasing magnitude of  $U$ .

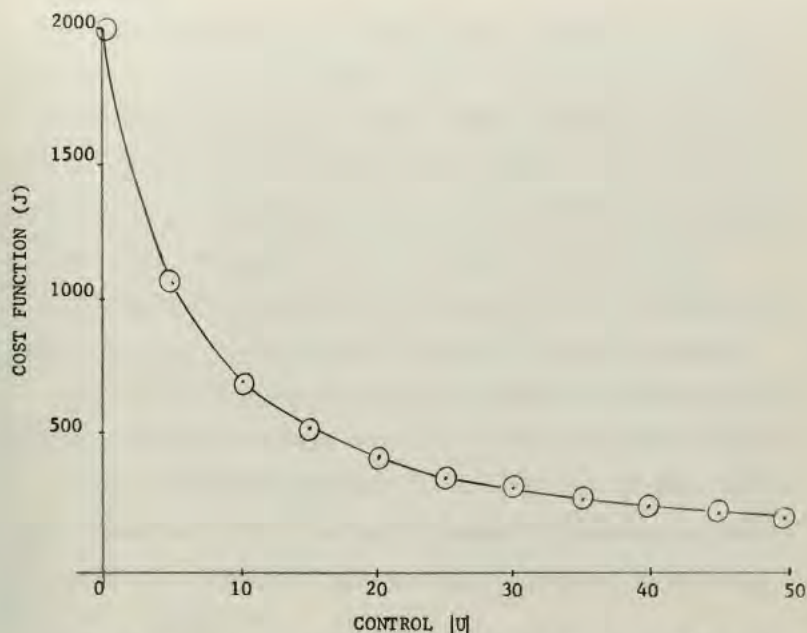


FIG. (6-3). A curve of Cost Function plotted against  $|U|$  showing the decrease in cost function (J) with the increase in  $|U|$ .

Through investigation of this system employing various values of  $U$ , it was found that the magnitude of the control effected both the magnitude and the frequency of the chatter. Additional graphical results may be found in Appendix IV. From an engineering point of view, it might be desirable to include a dead zone around zero in order to eliminate or reduce the chatter.

## 7. Conclusions.

In this investigation, the authors have inquired into and shown the feasibility of obtaining a numerical model for the optimal control of a non-linear system described by a first degree differential equation by means of the theory and procedures of Pontryagin's Maximum Principle as set forth by L. I. Rozonoer /1, 2, 3/. The investigation was limited to the "free right end" problem and to the single dimensional case.

The methods and procedures utilized may easily be extended to the n-dimensional case without any basic change in theory or procedure.

Various cost functions have been investigated and some conclusions verified by numerical solution utilizing the CDC 1604 digital computer.

Future investigation might well be in the area of second or third order systems and also in the area of computer programming techniques. The methods and programs utilized to obtain solutions were effective, but not very efficient.

The theories utilized here do yield solutions. However, other methods (Dynamic Programming for instance) might well be more effective and useful to the investigator, depending upon his specific needs.

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APPENDIX I

GRAPHS FOR THE UNCONTROLLED SYSTEM



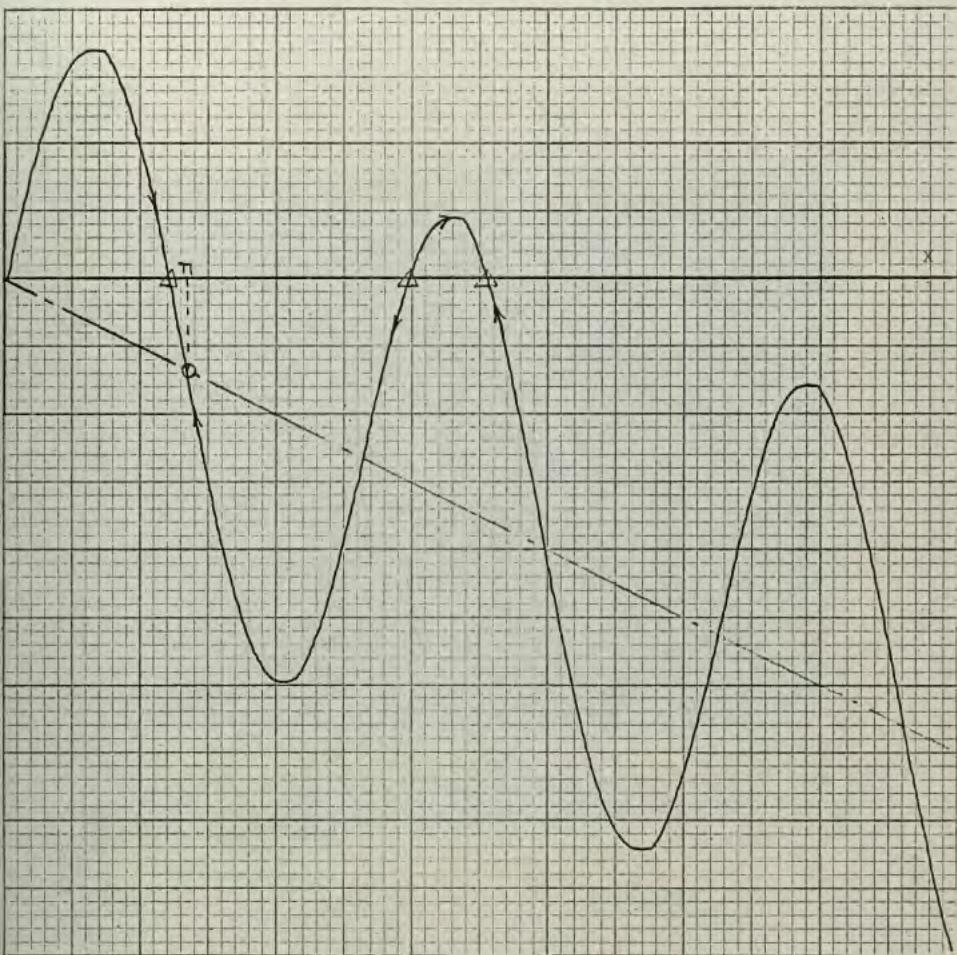


FIG. (I-1) Phase plane plot showing velocity as the ordinate against distance as the abscissa for the uncontrolled system having initial  $X$  equal to 20. This figure shows the graphical transformation of the abscissa from real time to distance. (Scale:  $Y-.05/\text{line}$ ;  $X-.236 \text{ rad}/\text{line}$ .)

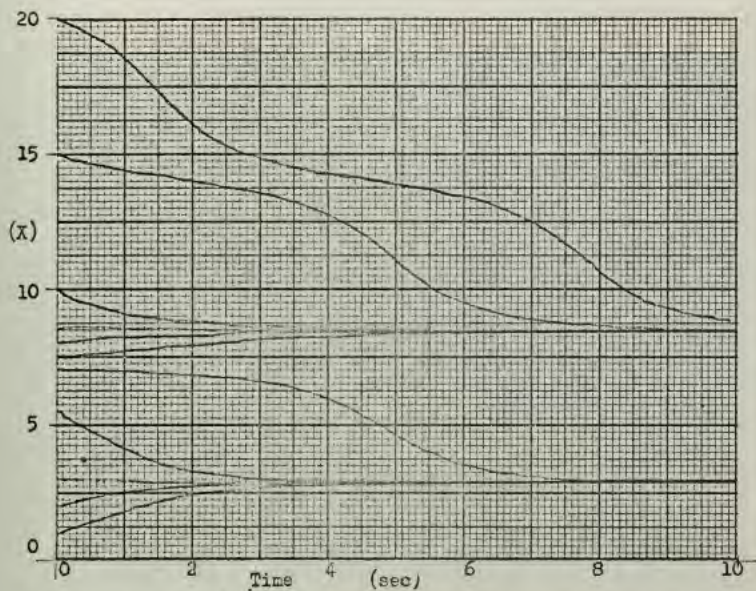


FIG. (I-2) Uncontrolled System Trajectory ( $X$ ) vs Time, for 11 positive values of  $X(0)$  showing the two stable equilibrium points,  $X = 2.85$ , and  $8.42$  for the positive system.

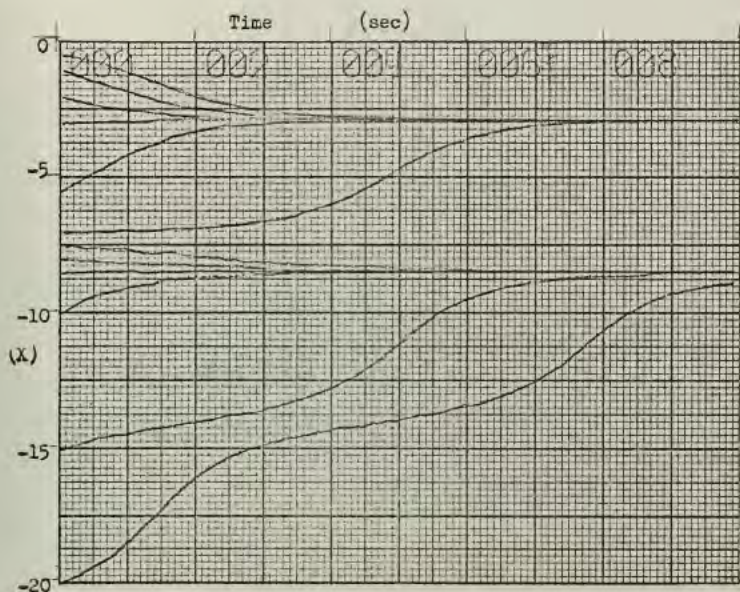


FIG. (I-3) Uncontrolled System Trajectory ( $X$ ) vs Time, for 11 negative values of  $X(0)$  showing the two stable equilibrium points,  $X = -2.85$ , and  $-8.52$  for the negative system.

APPENDIX II

GRAPHS FOR THE COST FUNCTION  $J = \int_0^T (X^2 + U^2) dt$



GRAPHS FOR THE COST FUNCTION  $J = \int_0^T (X^2 + U^2) dt$

<u>FIGURE NUMBER</u>	<u>DESCRIPTION OF GRAPH</u>	<u>COMMENTS</u>
II-1	J vs. P(0), X(0) = 20.5	Final X = 2.78
II-2	J vs. P(0), X(0) = 17.5	Final X = 2.77
II-3	" " = 15.0	" = 2.86
II-4	" " = 7.5	" = 1.57
II-5	" " = 5.0	" = 0.218
II-6	" " = 2.5	" = 0.024
II-7	" " = 1.0	" = 2.77
II-8	X vs. Time, X(0) = 20.0 (2 curves)	P*(0) = -128.0, (integer accuracy) J = 1156.01 and P*(0) = -127.4288 J = 1099.52
II-9	U vs. Time, X(0) = 20.0 (2 curves)	Same conditions as in II-8 above
II-10	X and U vs. Time, X(0) = 15.0	P*(0) = -93.82
II-11	X and U vs. Time, X(0) = 10.0	P*(0) = -41.96
II-12	X and U vs. Time, X(0) = 5.0	P*(0) = -14.7688
II-13	X vs. Time, X(0) = 20.0	Time interval = 20 sec.
II-14	U vs. Time, X(0) = 20.0	Time interval = 20 sec.

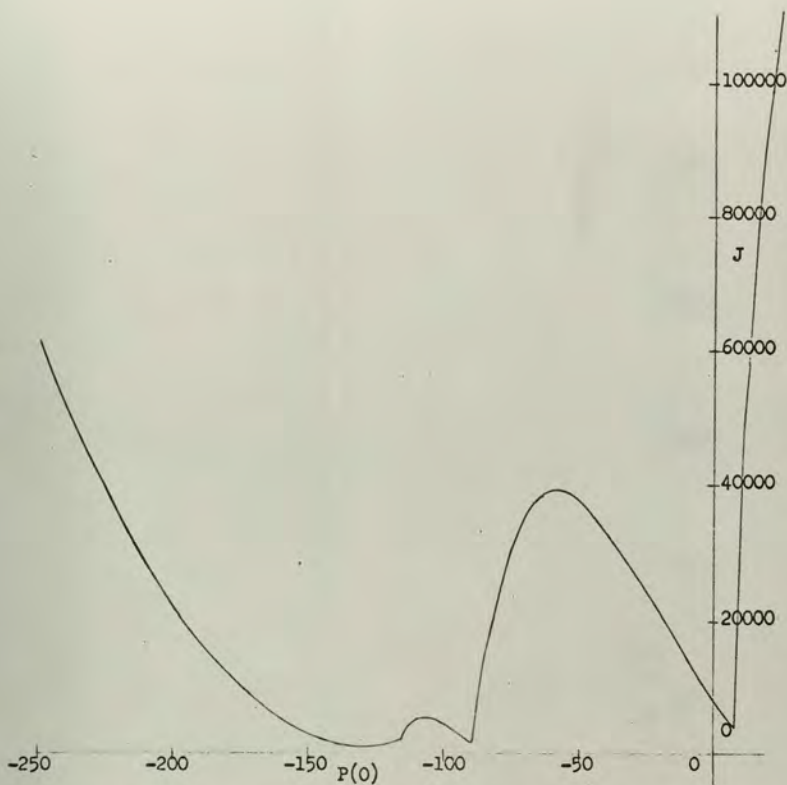


FIG. (II-1) Cost Function,  $J = \int_0^T (X^2 + U^2) dt$ , plotted as a function of initial value of the adjoint variable  $P$ , showing the true minimum and two local minima for the system having an initial  $X = 20$ .

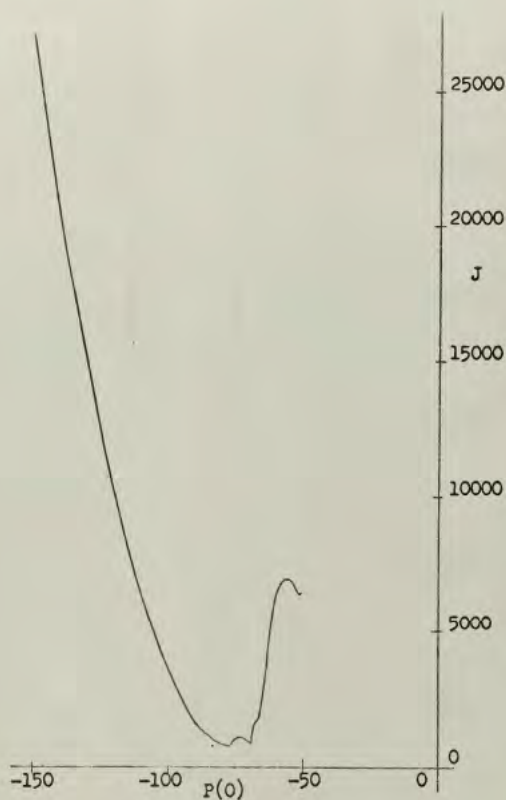


FIG. (II-2) Cost Function,  $J = \int_0^T (X^2 + U^2) dt$ , plotted as a function of initial value of the adjoint variable  $P$ , showing the true minimum and one local minimum for the system having an initial  $X = 17.5$ .

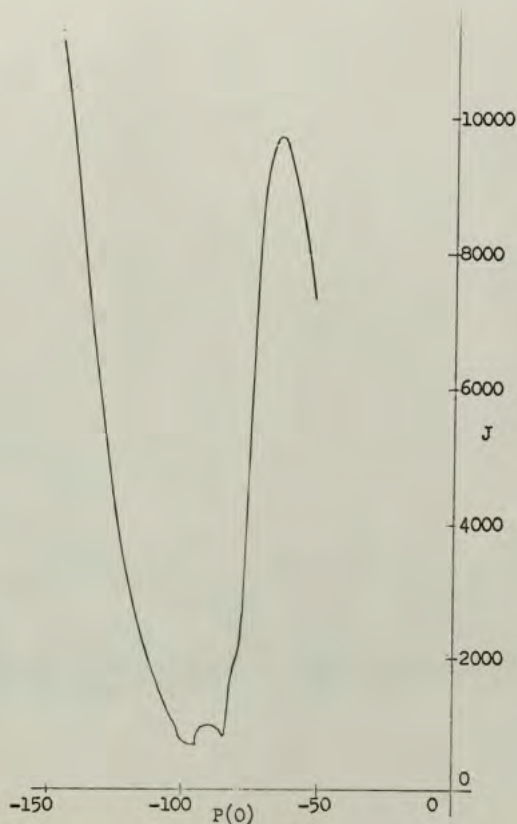


FIG. (II-3) Cost Function,  $J = \int_0^T (\dot{X}^2 + U^2) dt$ , plotted as a function of initial value of the adjoint variable  $P$ , showing the true minimum and one local minimum for the system having an initial  $X = 15$ .



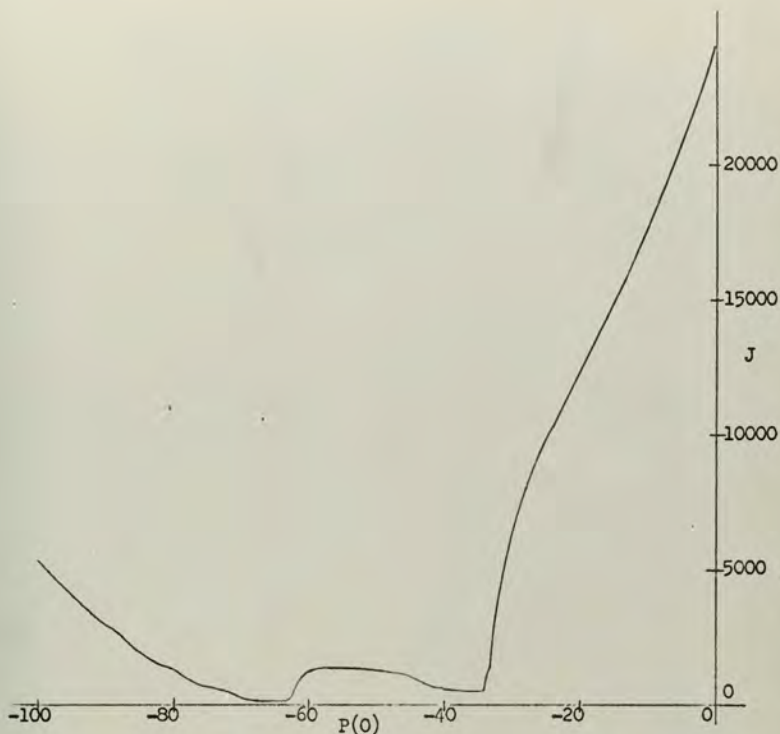


FIG. (II-4) Cost Function,  $J = \int_0^T (\dot{X}^2 + U^2) dt$ , plotted as a function of initial value of the adjoint variable  $P$ , showing the true minimum and two local minima for the system having an initial  $X = 7.5$ .

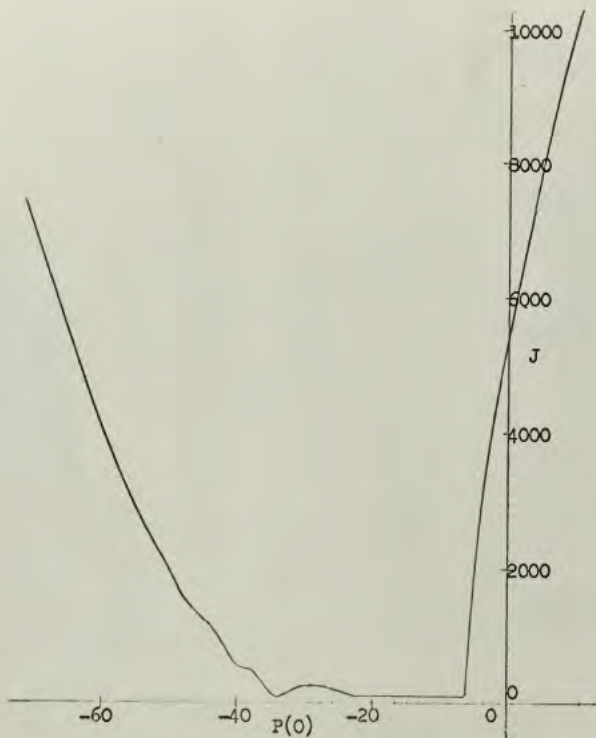


FIG. (II-6) Cost Function,  $J = \int_0^T (\dot{X}^2 + U^2) dt$ , plotted as a function of initial value of the adjoint variable  $P$ , showing the true minimum and two local minima for the system having an initial  $X = 2.5$ .

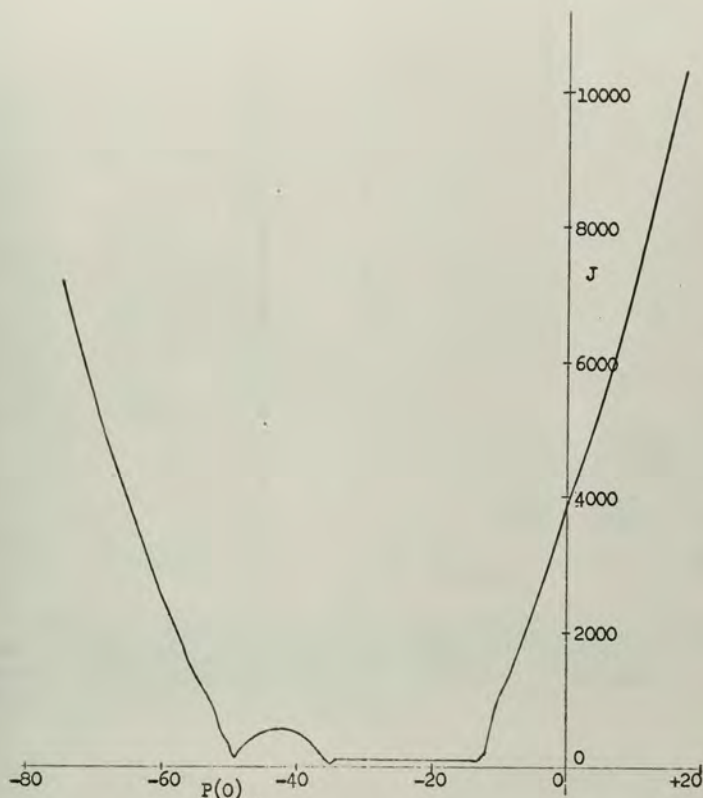


FIG. (II-7) Cost Function,  $J = \int_0^T (\dot{X}^2 + U^2) dt$ , plotted as a function of initial value of the adjoint variable  $P$ , showing the true minimum and two local minima for the system having an initial  $X = 1.0$ .

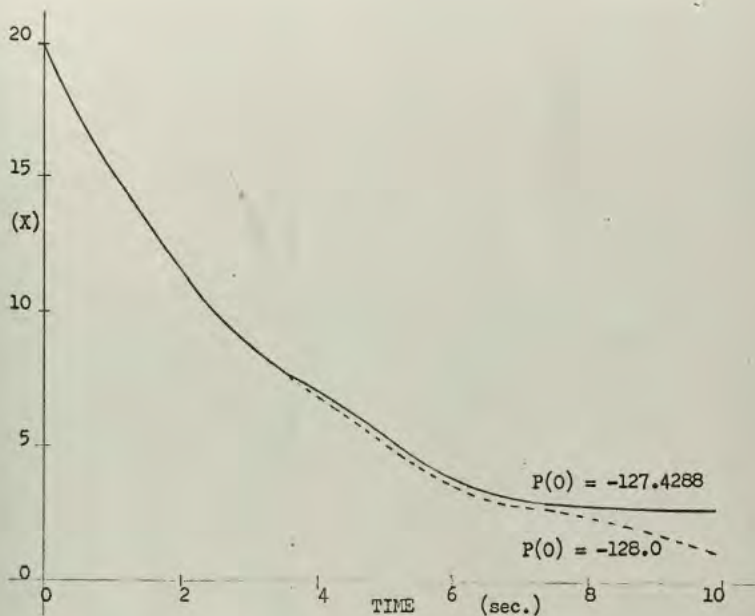


FIG. (II-8) Optimal Trajectory ( $X$ ) plotted against Time, for two values of  $P^*(0)$  showing the variation in trajectory as a function of the accuracy to which  $P^*(0)$  is determined; for the system having the Cost Function  $J = \int_0^T (X^2 + U^2) dt$  and initial  $X = 20$ .



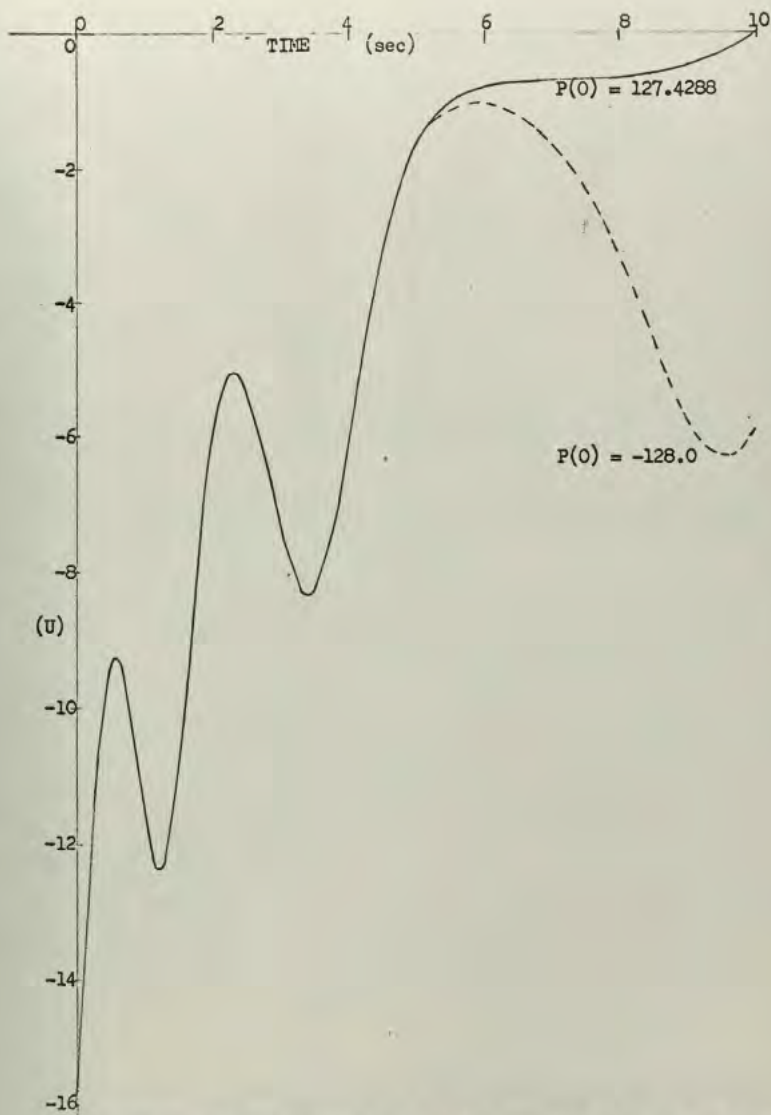


FIG. (II-9) Optimal Control ( $U$ ) plotted against Time, for two values of  $P^*(0)$  showing the variation in control as a function of the accuracy, to which  $P^*(0)$  is determined; for the system having the Cost Function  $J = \int_0^{\infty} (X^2 + U^2) dt$  and initial  $X = 20$ .

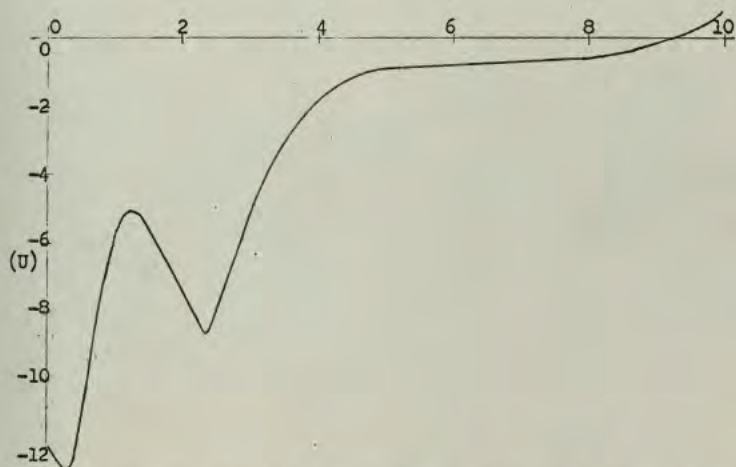
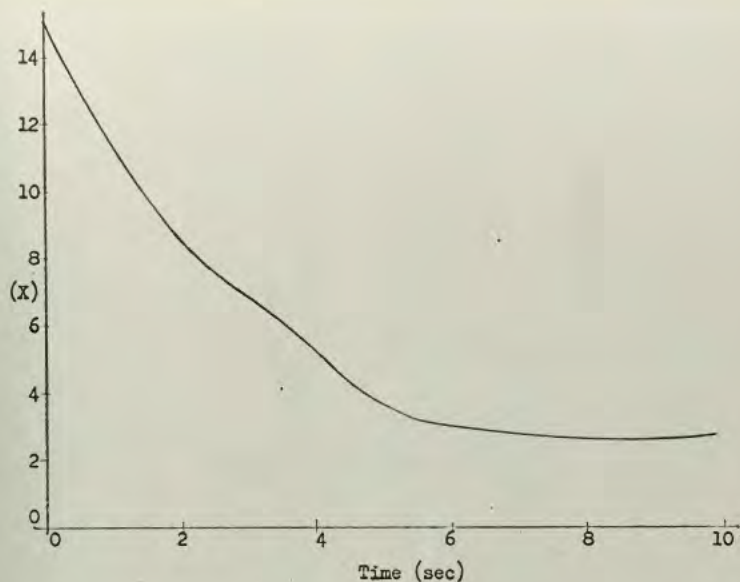


FIG. (II-10) Curves of Control (U) and Trajectory (X), plotted against Time, for the optimal system having the Cost Function  $J = \int_0^{\infty} (X^2 + U^2) dt$  and initial  $X = 15$ .

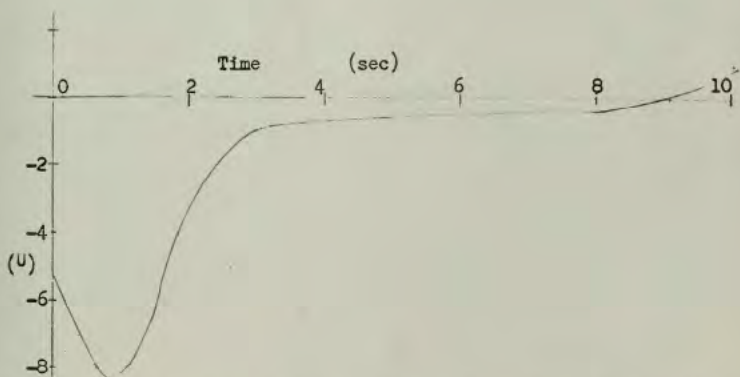
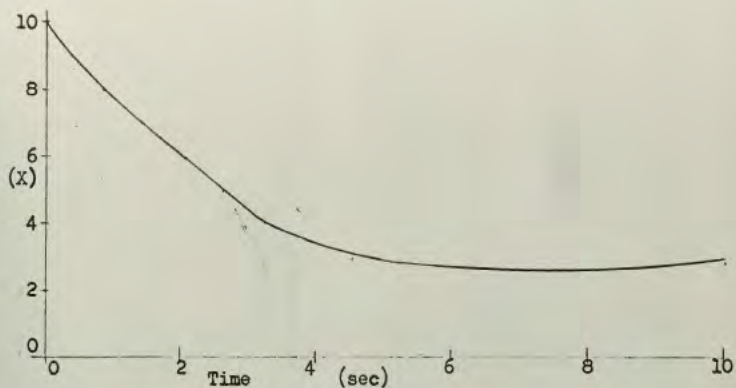


FIG. (II-11) Curves of Control ( $U$ ) and Trajectory ( $X$ ), plotted against Time, for the optimal system having the Cost Function  $J = \int_0^{\infty} (X^2 + U^2) dt$  and initial  $X = 10$ .

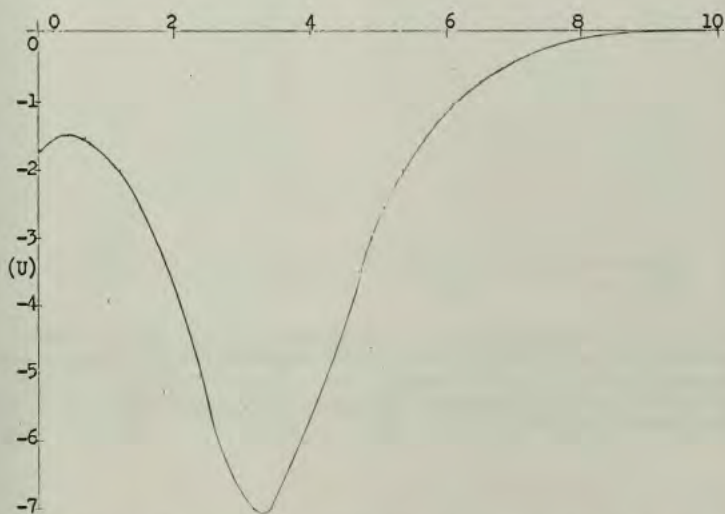
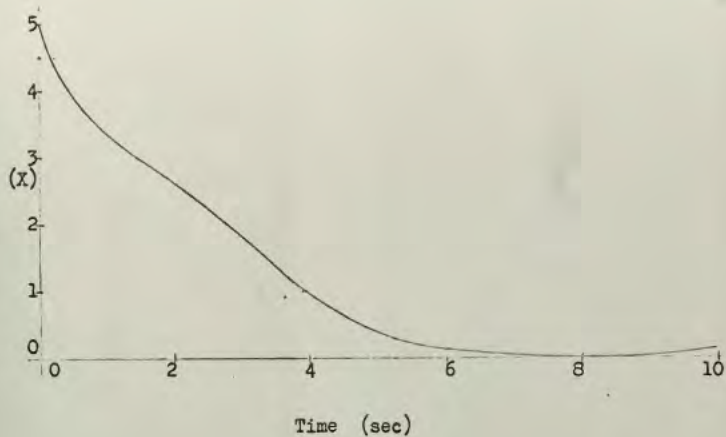


FIG. (II-12) Curves of Control (U) and Trajectory (X). plotted against Time, for the optimal system having the Cost Function  $J = \int_0^T (X^2 + U^2) dt$  and initial  $X = 5$ .



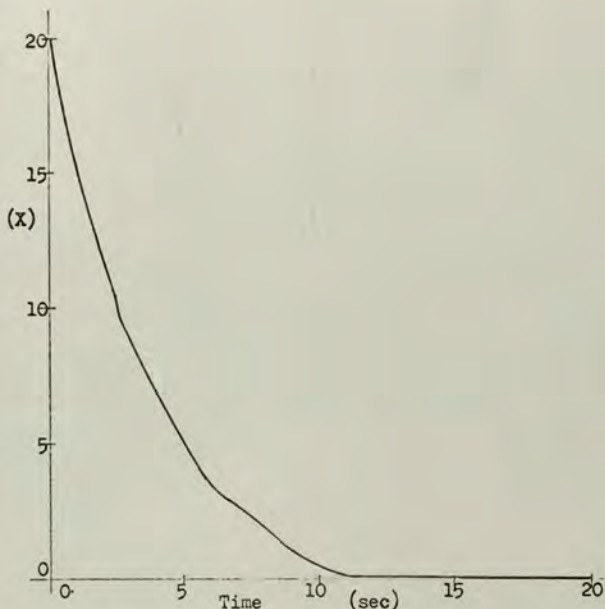


FIG. (II- 13) Optimal Trajectory ( $X$ ) plotted against Time, for the system where  $T$  final was extended to 20 seconds, having the Cost Function  $J = \int_0^T (X^2 + U^2) dt$  and initial  $X = 20.0$ . (Note when the time was extended from 10 to 20 seconds that the trajectory approached zero and was driven through the lower stable equilibrium point,  $X = 2.85$  by the optimum controller.)

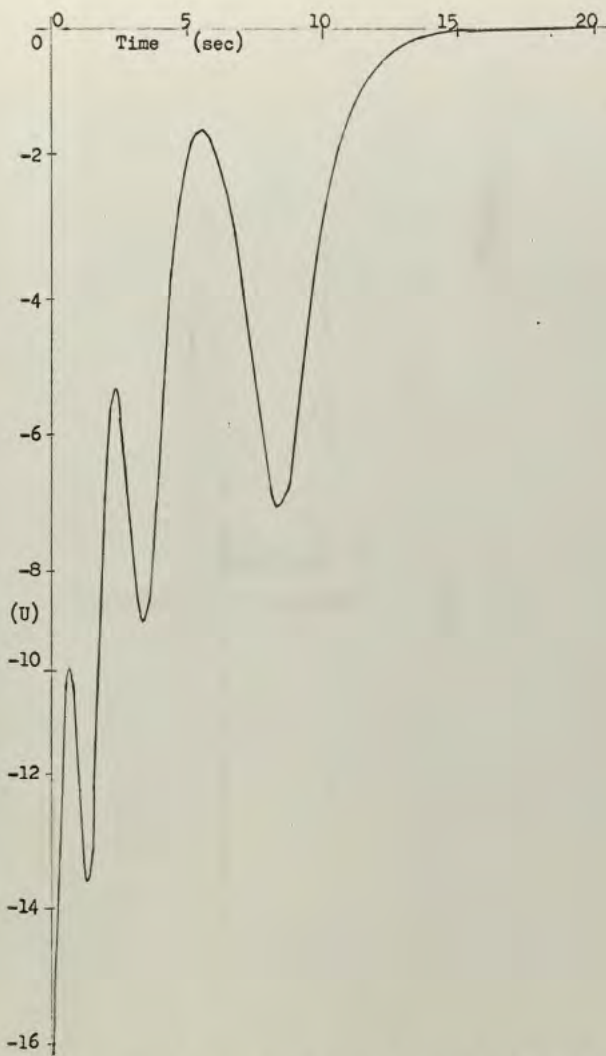


FIG. (II-14) Optimal Control ( $U$ ) plotted against Time, for the system where  $T$  final was extended to 20 seconds, having the Cost Function  $J = \int_0^T (X^2 + U^2) dt$  and initial  $X = 20$ .

### APPENDIX III

GRAPHS FOR THE COST FUNCTION  $J = \int_0^T U^2 dt$

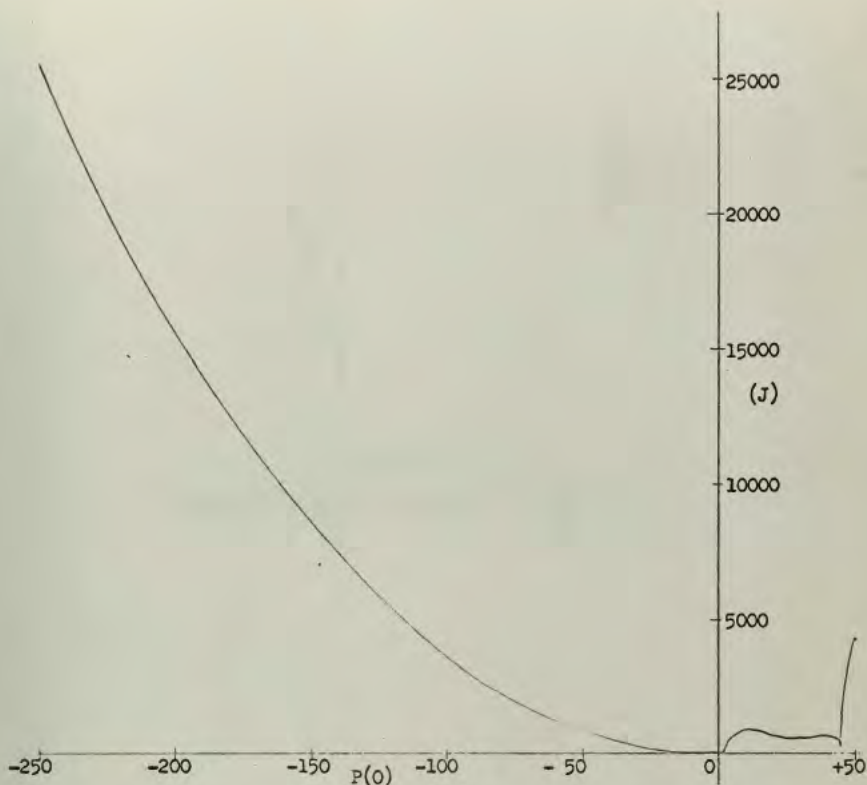


FIG. (III-1) Cost Function,  $J = \int_0^T (U^2) dt$ , plotted as a function of initial value of the adjoint variable  $P$ , showing the true minimum at  $P(0) = 0.0$  with  $J = 0.0$  for the system having an initial  $X = 20.0$ .

APPENDIX IV

GRAPHS FOR THE COST FUNCTION  $J = \int_0^T \dot{x}^2 dt$



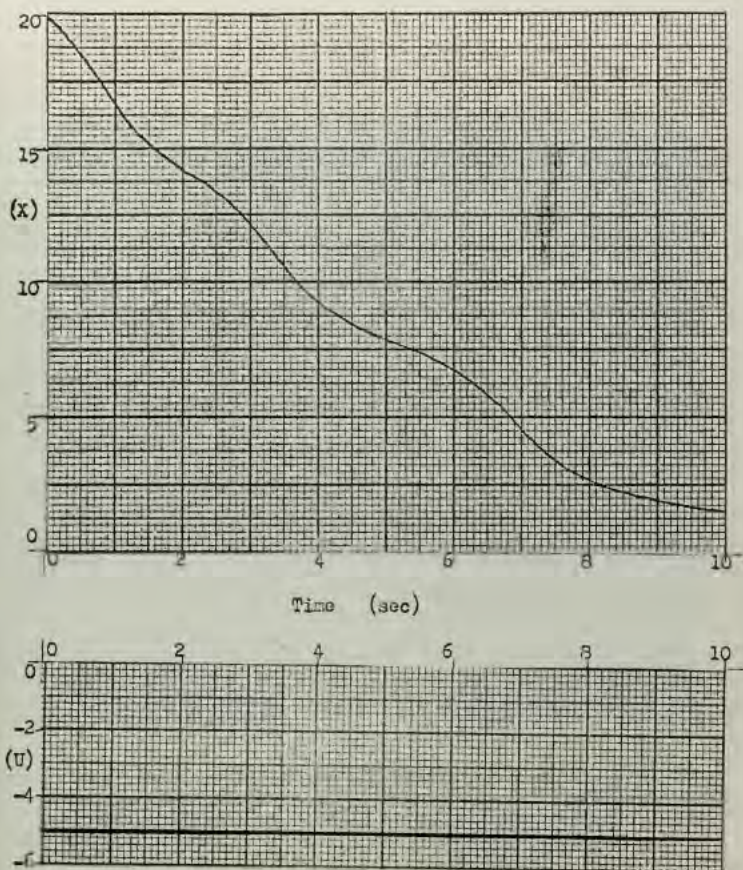


FIG. (IV-1) Curves of Control  $(U)$  and Trajectory  $(X)$ , plotted against Time, for the optimal system having the Cost Function  $J = \int_0^T (X^2) dt$  at initial  $X = 20$ , and  $|U|_{\max} = 5$ . (Note the absence of a chatter mode, as  $|U|_{\max}$  was not sufficient to drive the trajectory to zero in the allotted time.)

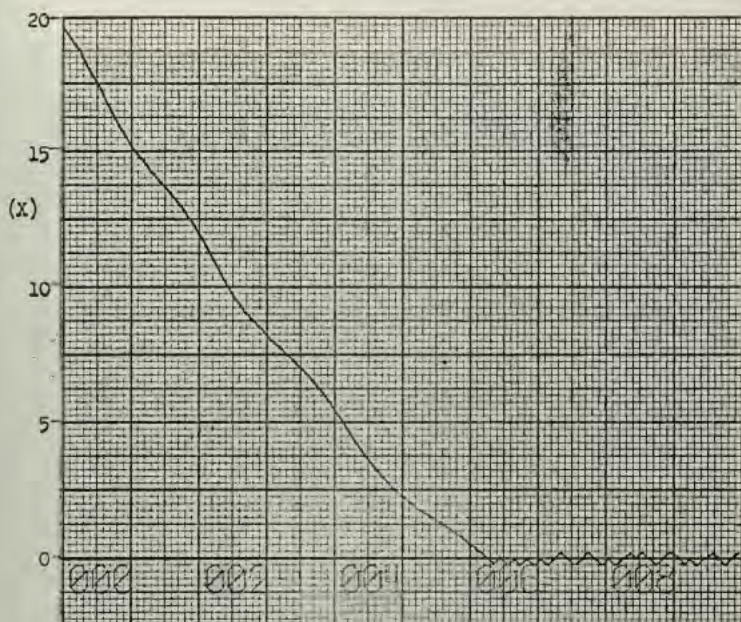


FIG. (IV-2) Optimal Trajectory  $(X)$  plotted against Time, for the system having the Cost Function  $J = \int_0^T (X^2) dt$  and initial  $X = 20$ . (Note the establishment of a chatter mode.)  $|U|_{\max} = 10$ .

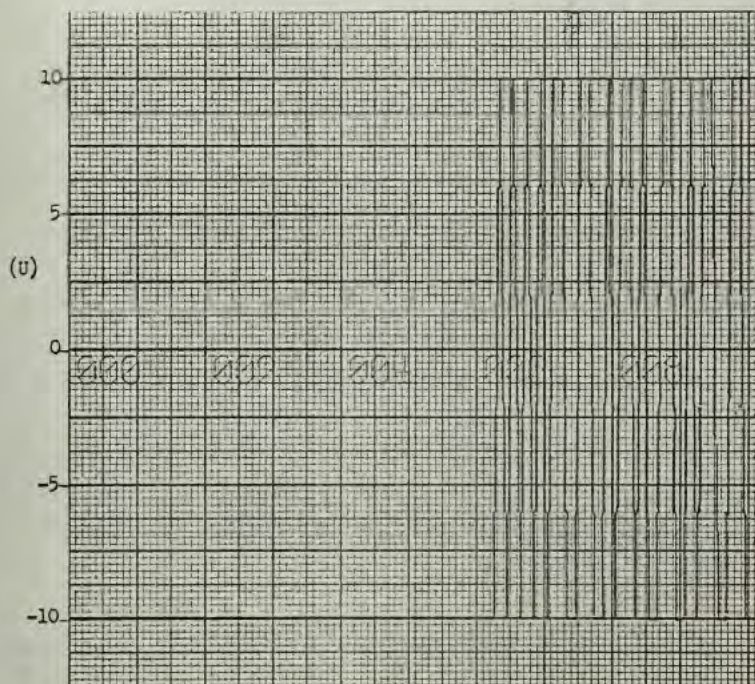


FIG.(IV-3) Optimal Control ( $U$ ) plotted against Time, for the system having the Cost Function  $J = \int_0^T (\dot{x}^2) dt$  and initial  $x = 20$ . (Note the establishment of a chatter mode.)  $|U|_{\max} = 10$ .



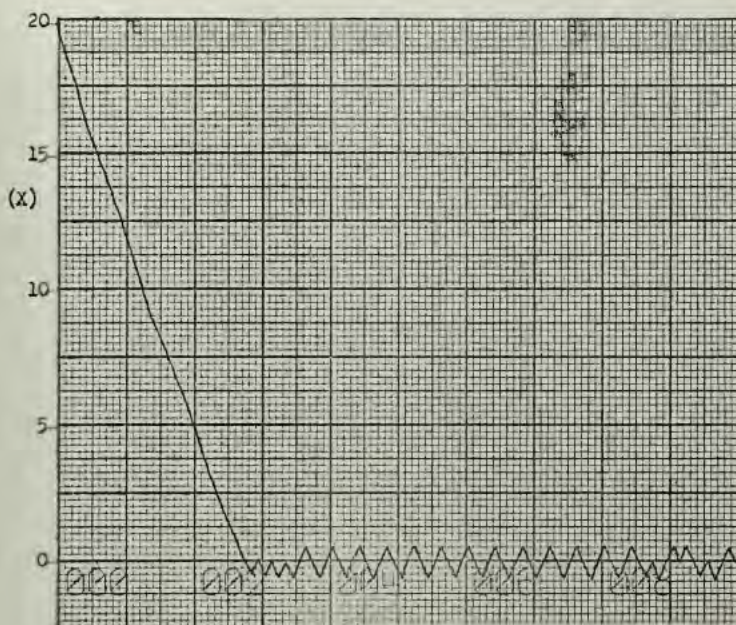


FIG. (IV-4) Optimal Trajectory  $X(t)$  plotted against Time, for the system having the Cost Function  $J = \int_0^T (X^2) dt$  and initial  $X = 20$ . (Note the establishment of a chatter mode.)  $|U|_{\max} = 25$ .

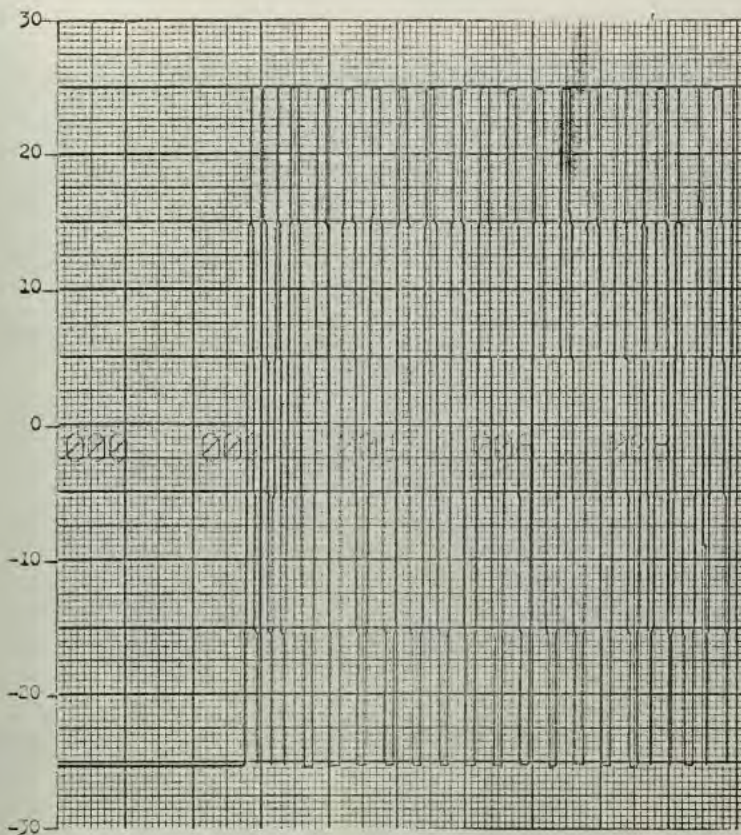


FIG. (IV-5) Optimal Control ( $U$ ) plotted against Time, for the system having the Cost Function  $J = \int_0^{\infty} U^2 dt$  and initial  $X = 20$ . (Note the establishment of a chatter mode.)  $|U|_{\max} = 25$ .



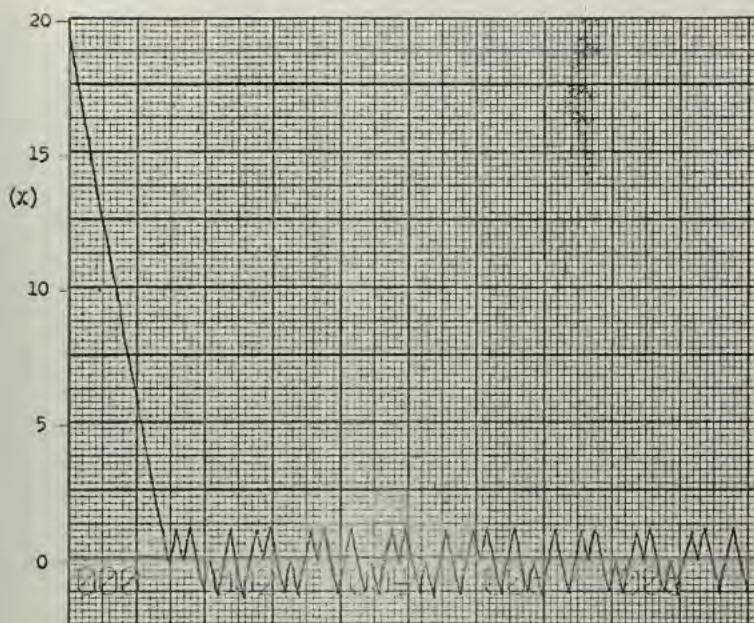


FIG. (IV-6) Optimal Trajectory  $(x)$  plotted against Time, for the system having the Cost Function  $J = \int_0^T (x^2) dt$  and initial  $x = 20$ . (Note the establishment of a chatter mode.)  $|U|_{\max} = 50$ .

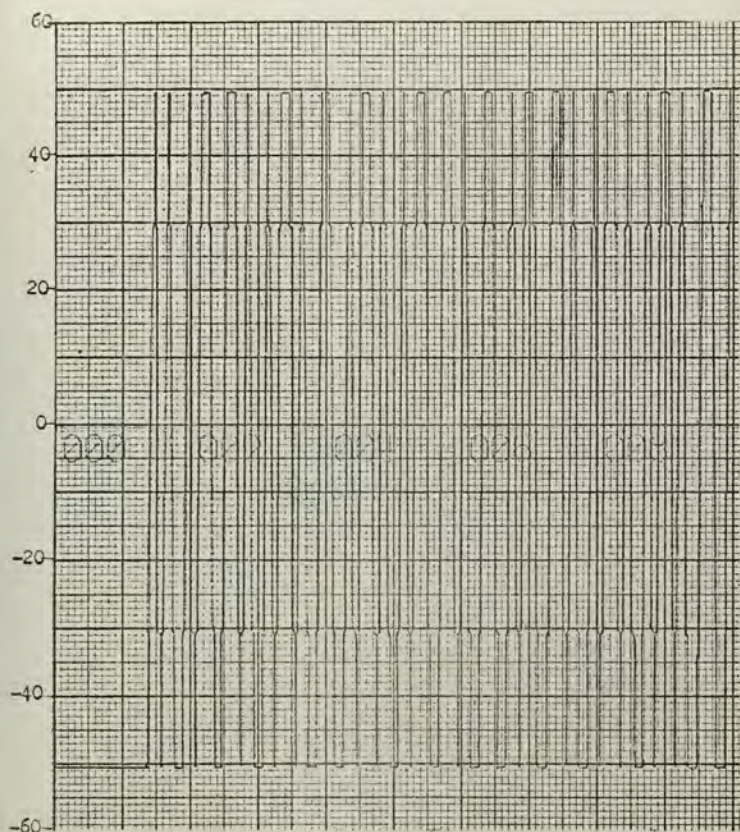


FIG. (IV-7) Optimal Control ( $u$ ) plotted against Time, for the system having the Cost Function  $J = \int_0^{\infty} u^2 dt$  and initial  $X = 20$ . (Note the establishment of a chatter mode.)  $|u|_{\max} = 50$ .

APPENDIX Va  
SUBROUTINE RUNGE-KUTTA

V.a. The Runge-Kutta method of evaluating differential equations.

There are many variants of the Runge-Kutta method, but the most widely used one is the following: given

$$y' = f(x, y)$$

$$y(x_n) = y_n$$

we compute in turn

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + h/2, y_n + k_1/2)$$

$$k_3 = h f(x_n + h/2, y_n + k_2/2)$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

This process may be described in geometric terms. At the point  $(x_n, y_n)$  we compute the slope  $(k_1/h)$  and using it we go one half step forward and examine the slope there. Using this new slope  $(k_2/h)$  we again start at  $(x_n, y_n)$ , go one half step forward, and again sample the slope. Using this latest slope  $(k_3/h)$  we again start at  $(x_n, y_n)$  but this time we go a full step forward where we examine the slope  $(k_4/h)$ . The four slopes are averaged, using weights  $1/6, 2/6, 2/6, 1/6$ , and using this average slope we make the final step from  $(x_n, y_n)$  to  $(x_{n+1}, y_{n+1})$ . If  $f(x, y)$  did not depend on  $y$ , then the averaging would produce Simpson's formula. The method has an error term proportional to  $h^5$ .

It is evident that the method throws away all old information and begins each complete step anew, and hence is hardly likely to be as efficient as methods which take advantage of old information. It is also evident that there is no check on whether the step size is too small or too large, though perhaps a study of the  $k_i$  might give a clue, this is not usually done.



The general spirit of the derivation is that the functions  $f(x,y)$  which are on the right-hand sides are all expanded in series in powers of  $h$  and the corresponding derivatives are equated to eliminate the lower powers of  $h$ .

The method as used in a subroutine for the investigations of this thesis is given below in Fortran language.

```

SUBROUTINE RKUTTA(N,T,X,DT)
  DIMENSION X(300), AK(4,300), XDOT(300), C(4)
  C(1)=0.0
  C(2)=0.5
  C(3)=0.5
  C(4)=1.0
  DO 4 I=1,4
    TC=T + C(I)*DT
    DO 2 J=1,N
      2 XC(J)=X(J) + C(I)*AK(I-1,J)
      CALL DERIV (TC, XC, XDOT, N)
      DO 4 J=1,N
        4 AK(I,J)=DT*XDOT(J)
      DO 3 J=1,N
        3 X(J)+X(J) + (AK(1,J) +2.*AK(3,J) + AK(4,J))/6.
      RETURN
    END

```



APPENDIX Vb

FORTRAN LANGUAGE PROGRAM TO OBTAIN TRAJECTORIES FOR THE  
UNCONTROLLED SYSTEM

```

..JOB GIBSON 9 JAN 1963 SPECIAL RUN
PROGRAM UNCON
DIMENSION X(30), X1(900), Y1(900), Y2(900), Y3(900), Y4(900)
1, Y5(900), Y6(900), Y7(900), Y8(900), Y9(900), Y10(900),
2 Y11(900), Y12(900)
NUMPTS = 0
READ 3, N, TC, TF, DT, (X(J), J=1, N)
N = NO. OF EQNS., TO = INITIAL VALUE, TF = FINAL VALUE
C DT = TIME STEP, X(J) = ARRAY OF DEPENDENT VARIABLES
C 3 FORMAT(I10/(8F10.0))
T = TO
1 NUMPTS = NUMPTS + 1
X1(NUMPTS) = T
Y1(NUMPTS) = X(1)
Y2(NUMPTS) = X(2)
Y3(NUMPTS) = X(3)
Y4(NUMPTS) = X(4)
Y5(NUMPTS) = X(5)
Y6(NUMPTS) = X(6)
Y7(NUMPTS) = X(7)
Y8(NUMPTS) = X(8)
Y9(NUMPTS) = X(9)
Y10(NUMPTS) = X(10)
Y11(NUMPTS) = X(11)
Y12(NUMPTS) = X(12)
10 IF(TF - T - DT) 10, 20, 20
CALL GRAPH (NUMPTS, X1, Y12, 8)
CALL GRAPH (NUMPTS, X1, Y11, 8)
CALL GRAPH (NUMPTS, X1, Y10, 8)
CALL GRAPH (NUMPTS, X1, Y9, 8)
CALL GRAPH (NUMPTS, X1, Y8, 8)
CALL GRAPH (NUMPTS, X1, Y7, 8)
CALL GRAPH (NUMPTS, X1, Y6, 8)
CALL GRAPH (NUMPTS, X1, Y5, 8)
CALL GRAPH (NUMPTS, X1, Y4, 8)
CALL GRAPH (NUMPTS, X1, Y3, 8)
CALL GRAPH (NUMPTS, X1, Y2, 8)
CALL GRAPH (NUMPTS, X1, Y1, 8)
104 FORMAT (12H0TRAJECTORY / (6E16.9))
PRINT 104, (Y1(I), I = 1, NUMPTS)
PRINT 104, (Y2(I), I = 1, NUMPTS)
PRINT 104, (Y3(I), I = 1, NUMPTS)
PRINT 104, (Y4(I), I = 1, NUMPTS)
PRINT 104, (Y5(I), I = 1, NUMPTS)
PRINT 104, (Y6(I), I = 1, NUMPTS)
PRINT 104, (Y7(I), I = 1, NUMPTS)
PRINT 104, (Y8(I), I = 1, NUMPTS)
PRINT 104, (Y9(I), I = 1, NUMPTS)
PRINT 104, (Y10(I), I = 1, NUMPTS)
PRINT 104, (Y11(I), I = 1, NUMPTS)
PRINT 104, (Y12(I), I = 1, NUMPTS)
PRINT 105, NUMPTS
105 FORMAT (10H0NUMPTS = I10)
STOP
20 CALL RKUTTA(N, T, X, DT)
T = T + DT
GO TO 1
END
SUBROUTINE RKUTTA(N, T, X, DT)
DIMENSION X(30), AK(4, 30), XDOT(30), XC(30), C(4)
C(1) = 0.0
C(2) = 0.5
C(3) = 0.5
C(4) = 1.0
DO 4 I = 1, 4
TC = T + C(I) * DT
DO 2 J = 1, N
2 XC(J) = X(J) + C(I) * AK(I - 1, J)
CALL DERIV(TC, XC, XDOT)
DO 4 J = 1, N

```



#### APPENDIX Vc

REPRESENTATIVE FORTRAN LANGUAGE PROGRAM TO OBTAIN COST  
FUNCTION AS A FUNCTION OF INITIAL P (ADJOINT VARIABLE)

```

..JOB8 GIBSON 9 JAN 1963 SPECIAL RUN MAX TIME 15 MIN
PROGRAM PJUX
DIMENSION X(950), XI(950), Y1(950)
READ 3, TO, TF, DT, XI, AL, AH, DA
C TO= INITIAL VALUE, TF= FINAL VALUE, DT= TIME STEP, XI= INITIAL X
C AH= HIGHEST INITIAL ADJOINT, AL= LOWEST INITIAL ADJOINT,
C DA= SPACING OF INITIAL ADJOINTS POINTS
3 FORMAT (8F10.0)
N= 900
C FORMS MATRIX OF INITIAL VALUES OF X
DO 300 I= 1, N, 3
X(I)= XI
C FORMS MATRIX OF P-ZERO
X(2)= AL
DO 301 I= 5, N, 3
301 X(I)= X(I-3) + DA
C ZEROIZES INITIAL VALUES OF COST FUNCTION
DO 302 I= 3, N, 3
302 X(I)= 0.0
T= TO
NUMPTS= N/3
L= 2
DO 12 J= 1, NUMPTS
12 X1(J)= X(L)
L= L+3
21 IF (TF-T) 10, 20, 20
10 L= 3
DO 11 J= 1, NUMPTS
11 Y1(J)= X(L)
L= L+3
DO 13 I= 1, NUMPTS
J= 2*(I-1) + 1
13 X(I)= X(J)
PRINT 101
101 FORMAT (43HOP-ZERO VS. COST FUNCTION FINAL X)
PRINT 102 (X1(I), Y1(I), X(I), I=1, NUMPTS)
102 FORMAT (F8.2, E16.9, E16.9)
CALL GRAPH (NUMPTS, X1, Y1, 8)
STOP
20 CALL RKUTTA (N,T,X,DT)
T= T+DT
GO TO 21
END
SUBROUTINE RKUTTA(N,T,X,DT)
DIMENSION X(950), AK(4,950), XDOT(950), XC(950), C(4)
C(1)=0.0
C(2)=0.5
C(3)=0.5
C(4)=1.0
DO 4 I=1,4
TC=T+C(I)*DT
DO 2 J=1,N
2 XC(J)=X(J) + C(I)*AK(I-1,J)
CALL DERIV (TC, XC, XDOT, N)
DO 4 J=1,N
4 AK(I,J)= DT*XDOT(J)
DO 3 J= 1,N
3 X(J)=X(J)+AK(1,J)+2.*AK(2,J)+2.*AK(3,J)+AK(4,J))/6.
RETURN
END
SUBROUTINE DERIV(T,X,XDOT,N)
DIMENSION XDOT(950), X(950)
DO 500 K=1,N,3
L=K+1
M=K+2
XDOT(K)= -0.1* X(K)+ SIN(X(K))+ .03125* X(L)
XDOT(L)= 0.1* X(L)- X(L)* COS(X(K))+ 2.0* X(K)
500 XDOT(M)= X(K)**2 + .015625* X(L)**2
END
END

```

The following are the values of the variables read in for this program:

TO = 0.0	AL = -250.0
TF = 10.0	AH = 50.0
DT = .1	DA = 1.0
XI = 20.0	



#### APPENDIX Vd

REPRESENTATIVE FORTRAN LANGUAGE PROGRAM USED TO COMPUTE  
THE OPTIMUM CONTROL POLICY AND TRAJECTORY HAVING THE  
 $P^*(0)$  FOR THE SYSTEM.

```

JOB ALMSTEDT MK2 MOD 1 1/20/63 MAX TIME 5 MIN X(0)= 20.0 OPT. TF(20)
PROGRAM MARK2 MOD 1
7 DIMENSION X(30), X1(10,201), U(10,201), CFCN(10),
71 XRAY(201), UNCLE(201), TIME(201)
X1= TRAJECTORIES, U= CONTROL EFFORT, CFCN= COST FUNCTION
READ 101, N, TO, TF, DT, X1, AL
N= TOTAL NUMBER OF EONS, TO = INITIAL TIME, TF = FINAL TIME
X1= INITIAL VALUE OF X, AL = INITIAL P VALUE
DO 1 I = 1, N, 3
1 X(I)= X1
READ 102, (X(I), I=2, N, 3)
DO 5 I = 3, N, 3
5 X(I)=0.0
T= TO
KK= 1
NN = N/3
30 IF(TF - T) 10, 20, 20
10 L= 3
DO 11 I= 1, NN
CFCN(I)= X(L)
11 L=L+3
TIME(I) = 0.0
DO 15 I = 2, KK
15 TIME(I)= TIME(I-1)+ DT
DO 16 I= 1, NN
PRINT 103, CFCN(I)
DO 17 K = 1, KK
XRAY (K) = X1(I, K)
17 UNCLE (K) = U(I, K)
PRINT 104, (XRAY (K), K = 1, KK)
PRINT 105, (UNCLE (K), K= 1, KK)
NPTS = KK
CALL GRAPH ( NPTS, TIME, XRAY, 8)
CALL GRAPH ( NPTS, TIME, UNCLE, 8)
16 CALL GRAPH ( NPTS, XRAY, UNCLE, 8)
STOP
20 L= 1
DO 21 I= 1, NN
X1(I, KK) = X(L)
U(I, KK) = X(L+1)/8.0
21 L= L+3
KK= KK+ 1
CALL RKUTTA(N, T, X, DT)
T= T+ DT
GO TO 30
101 FORMAT ( 110, 7F10.3)
102 FORMAT(8F10.6)
103 FORMAT(17HOCOST FUNCTION = E16.9))
104 FORMAT(12HOTRAJECTORY /(6E16.9)
105 FORMAT(16HOCNTRLEFFORT /(6E16.9))
END
SUBROUTINE RKUTTA(N, T, X, DT)
DIMENSION X(601), AK(4,601), XDOT(601), XC(601), C(4)
C(1)=0.0
C(2)=0.5
C(3)=0.5
C(4)=1.0
DO 4 I=1,4
TC=T+C(I)*DT
DO 2 J=1, N
2 XC(J) =X(J) + C(I)*AK(I-1, J)
CALL DERIV (TC, XC, XDOT, N)
DO 4 J=1, N
4 AK(I, J) = DT*XDOT(J)
DO 3 J = 1, N
3 X(J)=X(J)+(AK(1, J)+2.*AK(2, J)+2.*AK(3, J)+AK(4, J))/6.
RETURN
END
SUBROUTINE DERIV(T, X, XDOT, N)
DIMENSION XDOT(601), X(601)

```

```

DO 500 K=1,N,3
L=K+1
M=K+2
XDOT(K) = -0.1* X(K)+ SIN(X(K))+ .03125* X(L)
XDOT(L) = 0.1* X(L) - X(L)* COS(X(K))+ 2.0* X(K)
500 XDOT(M) = X(K)**2 + .015625* X(L)**2
END
END

          90.0          20.0          0.1          20.0          0.0
128.955976

2 TRAJECTORY X VS TIME          P-ZERO = 20.0 NO 1 TF(20)          10 08
2 GIBSON AND ALMSTEDT          JAN 1963 MK2
0

2 CONTROL FUNCTION U VS TIME          NO 1          10 08
2 GIBSON AND ALMSTEDT          JAN 1963 MK2          TF(20)
0

2 INITIAL CONTROL VS. TRAJECTORY OPTIMAL SYSTEM TF(20)          10 08
2 GIBSON AND ALMSTEDT          JAN 1960 MK2
0

```

02

02

02

#### APPENDIX Ve

FORTRAN LANGUAGE PROGRAM UTILIZED IN THE EVALUATION  
OF THE TRAJECTORIES AND CONTROL POLICIES FOR THE  
SYSTEM HAVING  $\int (x^2) dt$  AS THE COST FUNCTION.

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..JOB GIBSON 22 JANUARY 1963 MK4MCD0 ALT1
PROGRAM MK4 MCD0 ALT 1
C THIS PROGRAM USES INTERGAL X SQUARE AS THE COST FUNCTION
DIMENSION X(30), XRAY(15,500), U(15), UNCLE(15,500), TIME(500),
Y1(101), Y2(101), V(15)
101 READ 101, NN, TO, TF, DT, XI, (U(I), I=1, NN)
101 FORMAT (18/ 15F5.0)
N = 2* NN
C FORM X-MATRIX
DO 1 I=1, NN
1 X(I)=XI
N1=NN+1
DO 2 I= N1, N
2 X(I)= 0.0
T= TO
NPTS= 0
27 IF(TF - T) 10, 20, 20
20 NPTS = NPTS + 1
IF ( 1 - NPTS) 21, 26, 26
21 CALL RKUTTA (N, T, X, DT, U, V)
T = T + DT
26 TIME(NPTS) = T
DO 28 I= 1, NN
IF ( X(I)) 23, 24, 25
23 UNCLE (I, NPTS) = -U(I)
GO TO 22
24 UNCLE (I, NPTS) = 0.0
GO TO 22
25 UNCLE (I, NPTS) = U(I)
22 XRAY (I, NPTS) = X(I)
28 V(I) = UNCLE(I, NPTS)
GO TO 27
10 DO 11 I= 1, NN
PRINT 111, X(I+NN)
111 FORMAT (17HOCOST FUNCTION = E16.9)
PRINT 112, (XRAY (I, J), J=1, NPTS)
112 FORMAT (12HOTRAJECTORY / (6E16.9))
PRINT 113 (UNCLE (I, J), J= 1, NPTS)
113 FORMAT (16HOCNTR0L EFFORT / (6E16.9))
DO 12 K=1, NPTS
Y1(K) = XRAY(I, K)
12 Y2(K) = UNCLE(I, K)
CALL GRAPH (NPTS, TIME, Y1, 8)
11 CALL GRAPH (NPTS, TIME, Y2, 8)
PRINT 114, NPTS
114 FORMAT ( 8HONPTS = I5)
STOP
END
SUBROUTINE RKUTTA (N, T, X, DT, U, V)
DIMENSION X(30), AK(4,30), XDOT(30), XC(30), C(4), V(15), U(30)
C(1)=0.0
C(2)=0.5
C(3)=0.5
C(4)=1.0
DO 4 I= 1, 4
TC = T + C(I)*DT
DO 2 J=1, N
2 XC(J) = X(J) + C(I)*AK(I-1, J)
CALL DERIV ( XC, TC, XDOT, N, U, V)
DO 4 J= 1, N
4 AK(I, J) = DT*XDOT(J)
DO 3 J= 1, N
3 X(J)=X(J)+(AK(1, J)+2.*AK(2, J)+2.*AK(3, J)+AK(4, J))/6.
RETURN
END
SUBROUTINE DERIV (X, T, XDOT, N, U, V)
DIMENSION XDOT(30), X(30), U(15), V(15)
NN = N/2
N1 = NN + 1
DO 1 I= 1, NN
1 XDOT(I) = -.1*X(I) + SIN(X(I)) +.25* V(I)
DO 2 I= N1, N
2 XDOT (I)= X(I-NN)**2
END
END

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0.0 10.0.1 20.0 -5.0 -10.-15. -20. -25. -30. -35. -40. -45. -50.



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Solution of a problem of optimal control



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